Higher Order Mixed Finite Elements for Maxwell's Equations

Math Open House 2023

Archana Arya
Ph.D. Student in Mathematics
November 25, 2023, Saturday

DEPARTMENT OF MATHEMATICS

Overview

- Computational electromagnetics entails numerical solution of Maxwell's equations and has been one of the foundational pillars of modern electrical engineering.
- We shall demonstrate higher order, structure preserving finite element methods for the solution of problems modelled using Maxwell's equations.
  - We discuss the proof idea for existence of solution for the weak formulation.
  - Our finite elements spaces shall be drawn from a de Rham sequence of conforming finite dimensional polynomial function spaces.
  - Our time discretization schemes will be Backward Euler and Crank-Nicholson.
- At last, we shall demonstrate some computational results using linear and quadratic finite elements.

We demonstrate our results for the following system of Maxwell's equations:

$$\begin{aligned} \frac{\partial p}{\partial t} + \nabla \cdot \varepsilon E &= f_p \text{ in } \Omega \times (0,T], \\ \nabla p + \varepsilon \frac{\partial E}{\partial t} - \nabla \times H &= f_E \text{ in } \Omega \times (0,T], \\ \mu \frac{\partial H}{\partial t} + \nabla \times E &= f_H \text{ in } \Omega \times (0,T], \end{aligned} \tag{1a}$$

where $\Omega \subset \mathbb{R}^2/\mathbb{R}^3$ is a domain with Lipschitz boundary $\partial\Omega$ and with the following homogeneous boundary conditions:

$$p = 0, E \times n = 0, H \cdot n = 0 \text{ on } \partial\Omega \times (0,T], \tag{1b}$$

where n is the unit outward normal to $\partial\Omega$, and with the following initial conditions:

$$p(x,0) = p_0(x), E(x,0) = E_0(x), \text{ and } H(x,0) = H_0(x) \text{ for } x \in \Omega. \tag{1c}$$

In these equations, $E(x,t)$ and $H(x,t)$ denote the electric and magnetic fields, respectively, and $p(x,t)$ is a physically fictitious electric pressure. The material parameters ε and μ denotes electric permittivity and magnetic permeability respectively. Finally, we shall assume that the initial conditions provided satisfy:

$$\nabla \cdot (\varepsilon E_0) = p_0, \text{ and } \nabla \cdot (\mu H_0) = 0 \text{ in } \Omega.$$

Weak Formulation

For given boundary conditions, find $(p, E, H) \in \mathring{H}^1_{\varepsilon^{-1}}(\Omega) \times \mathring{H}_\varepsilon(\mathbf{curl}; \Omega) \times \mathring{H}_\mu(\mathbf{div}; \Omega)$:

$$\langle \frac{\partial p}{\partial t}, \tilde{p} \rangle - \langle \varepsilon E, \nabla \tilde{p} \rangle = \langle f_p, \tilde{p} \rangle, \qquad \tilde{p} \in \mathring{H}^1_{\varepsilon^{-1}}(\Omega), \tag{2a}$$

$$\langle \nabla p, \widetilde{E} \rangle + \langle \varepsilon \frac{\partial E}{\partial t}, \widetilde{E} \rangle - \langle H, \nabla \times \widetilde{E} \rangle = \langle f_E, \widetilde{E} \rangle, \quad \widetilde{E} \in \mathring{H}_\varepsilon(\mathbf{curl}; \Omega), \tag{2b}$$

$$\langle \mu \frac{\partial H}{\partial t}, \widetilde{H} \rangle + \langle \nabla \times E, \widetilde{H} \rangle, = \langle f_H, \widetilde{H} \rangle, \quad \widetilde{H} \in \mathring{H}_\mu(\mathbf{div}; \Omega), \tag{2c}$$

for $t \in (0, T]$ with given initial conditions.

For given boundary conditions, find $(p, E, H) \in \mathring{H}^1_{\varepsilon^{-1}}(\Omega) \times \mathring{H}_\varepsilon(\mathbf{curl}; \Omega) \times \mathring{H}_\mu(\mathbf{div}; \Omega)$:

$$\langle \frac{\partial p}{\partial t}, \tilde{p} \rangle - \langle \varepsilon E, \nabla \tilde{p} \rangle = \langle f_p, \tilde{p} \rangle, \qquad \tilde{p} \in \mathring{H}^1_{\varepsilon^{-1}}(\Omega), \tag{2a}$$

$$\langle \nabla p, \widetilde{E} \rangle + \langle \varepsilon \frac{\partial E}{\partial t}, \widetilde{E} \rangle - \langle H, \nabla \times \widetilde{E} \rangle = \langle f_E, \widetilde{E} \rangle, \quad \widetilde{E} \in \mathring{H}_\varepsilon(\mathbf{curl}; \Omega), \tag{2b}$$

$$\langle \mu \frac{\partial H}{\partial t}, \widetilde{H} \rangle + \langle \nabla \times E, \widetilde{H} \rangle, = \langle f_H, \widetilde{H} \rangle, \quad \widetilde{H} \in \mathring{H}_\mu(\mathrm{div}; \Omega), \tag{2c}$$

for $t \in (0, T]$ with given initial conditions.

de Rham Complex

- Vector Calculus Version

$$\text{Scalar functions} \underset{-\,\mathrm{div}}{\overset{\mathrm{grad}}{\rightleftarrows}} \text{Vector fields} \underset{\mathrm{curl}}{\overset{\mathrm{curl}}{\rightleftarrows}} \text{Vector fields} \underset{-\,\mathrm{grad}}{\overset{\mathrm{div}}{\rightleftarrows}} \text{Density functions}$$

- Functional Analysis Version

$$\mathring{H}^1_{\varepsilon^{-1}}(\Omega) \underset{-\,\mathrm{div}}{\overset{\mathrm{grad}}{\rightleftarrows}} \mathring{H}_\varepsilon(\mathrm{curl}, \Omega) \underset{\mathrm{curl}}{\overset{\mathrm{curl}}{\rightleftarrows}} \mathring{H}_\mu(\mathrm{div}, \Omega) \underset{\mathrm{grad}}{\overset{-\,\mathrm{div}}{\rightleftarrows}} L^2(\Omega)$$

Energy

Energy of the Maxwell's equations is defined to be $\|p\|_{\varepsilon^{-1}}^2 + \|E\|_{\varepsilon}^2 + \|H\|_{\mu}^2$.

Theorem (Energy Estimate)

Let $f_p \in L^1[0,T] \times L^2_{\varepsilon^{-1}}(\Omega)$, $f_E \in L^1[0,T] \times L^2_{\varepsilon^{-1}}(\Omega)$, and $f_H \in L^1[0,T] \times L^2_{\mu^{-1}}(\Omega)$. Then the solution (p, E, H) of Equations (2a) to (2c) with initial conditions as in Equation (1c) and assuming sufficient regularity with $p \in C^1[0,T] \times \mathring{H}^1_{\varepsilon^{-1}}(\Omega),\ E \in C^1[0,T] \times \mathring{H}_{\varepsilon}(\mathbf{curl};\Omega)$, and $H \in C^1[0,T] \times \mathring{H}_{\mu}(\mathrm{div};\Omega)$ satisfies:

$$\|p\|_{\varepsilon^{-1}} + \|E\|_{\varepsilon} + \|H\|_{\mu} \leq \sqrt{3}\Big[\|p_0\|_{\varepsilon^{-1}} + \|E_0\|_{\varepsilon} + \|H_0\|_{\mu} + \|f_p\|_{L^1[0,T]\times L^2_{\varepsilon^{-1}}(\Omega)} + \|f_E\|_{L^1[0,T]\times L^2_{\varepsilon^{-1}}(\Omega)} + \|f_H\|_{L^1[0,T]\times L^2_{\mu^{-1}}(\Omega)}\Big].$$

Proof Idea

- We start our proof by choosing $\tilde{p} = \varepsilon^{-1} p, \widetilde{E} = E$ and $\widetilde{H} = H$ in Equations (2a) to (2c) shown below:

$$\langle \frac{\partial p}{\partial t}, \tilde{p} \rangle - \langle \varepsilon E, \nabla \tilde{p} \rangle = \langle f_p, \tilde{p} \rangle, \qquad \tilde{p} \in \mathring{H}^1_{\varepsilon^{-1}}(\Omega), \tag{2a}$$

$$\langle \nabla p, \widetilde{E} \rangle + \langle \varepsilon \frac{\partial E}{\partial t}, \widetilde{E} \rangle - \langle H, \nabla \times \widetilde{E} \rangle = \langle f_E, \widetilde{E} \rangle, \quad \widetilde{E} \in \mathring{H}_\varepsilon(\mathbf{curl}; \Omega), \tag{2b}$$

$$\langle \mu \frac{\partial H}{\partial t}, \widetilde{H} \rangle + \langle \nabla \times E, \widetilde{H} \rangle, = \langle f_H, \widetilde{H} \rangle, \quad \widetilde{H} \in \mathring{H}_\mu(\mathbf{div}; \Omega), \tag{2c}$$

- Adding the resultant equations and using the properties of the inner product, we get:

$$\langle \frac{\partial p}{\partial t}, \varepsilon^{-1} p \rangle + \langle \varepsilon \frac{\partial E}{\partial t}, E \rangle + \langle \mu \frac{\partial H}{\partial t}, H \rangle = \langle f_p, \varepsilon^{-1} p \rangle + \langle f_E, E \rangle + \langle f_H, H \rangle.$$

Proof Idea

- We start our proof by choosing $\tilde{p} = \varepsilon^{-1} p, \widetilde{E} = E$ and $\widetilde{H} = H$ in Equations (2a) to (2c) shown below:

$$\langle \frac{\partial p}{\partial t}, \tilde{p} \rangle - \langle \varepsilon E, \nabla \tilde{p} \rangle = \langle f_p, \tilde{p} \rangle, \qquad \tilde{p} \in \mathring{H}^1_{\varepsilon^{-1}}(\Omega), \tag{2a}$$

$$\langle \nabla p, \widetilde{E} \rangle + \langle \varepsilon \frac{\partial E}{\partial t}, \widetilde{E} \rangle - \langle H, \nabla \times \widetilde{E} \rangle = \langle f_E, \widetilde{E} \rangle, \quad \widetilde{E} \in \mathring{H}_\varepsilon(\mathbf{curl}; \Omega), \tag{2b}$$

$$\langle \mu \frac{\partial H}{\partial t}, \widetilde{H} \rangle + \langle \nabla \times E, \widetilde{H} \rangle, = \langle f_H, \widetilde{H} \rangle, \quad \widetilde{H} \in \mathring{H}_\mu(\mathbf{div}; \Omega), \tag{2c}$$

- Adding the resultant equations and using the properties of the inner product, we get:

$$\langle \frac{\partial p}{\partial t}, \varepsilon^{-1} p \rangle + \langle \varepsilon \frac{\partial E}{\partial t}, E \rangle + \langle \mu \frac{\partial H}{\partial t}, H \rangle = \langle f_p, \varepsilon^{-1} p \rangle + \langle f_E, E \rangle + \langle f_H, H \rangle.$$

- Consider $\frac{d}{dt} \|p\|^2_{\varepsilon^{-1}} = 2 \langle \frac{\partial p}{\partial t}, \varepsilon^{-1} p \rangle$, $\frac{d}{dt} \|E\|^2_\varepsilon = 2 \langle \varepsilon \frac{\partial E}{\partial t}, E \rangle$, and $\frac{d}{dt} \|H\|^2_\mu = 2 \langle \mu \frac{\partial H}{\partial t}, H \rangle$.
- Using these, Cauchy-Schwarz inequality, integrating w.r.t. 't', and further using Gronwall-OuLang inequality we obtain:

$$\sqrt{\|p\|^2_{\varepsilon^{-1}} + \|E\|^2_\varepsilon + \|H\|^2_\mu} \leq \sqrt{\|p_0\|^2_{\varepsilon^{-1}} + \|E_0\|^2_\varepsilon + \|H_0\|^2_\mu} + \int_0^T \left(\|f_p\|_{\varepsilon^{-1}} + \|f_E\|_{\varepsilon^{-1}} + \|f_H\|_{\mu^{-1}} \right) ds.$$

- Finally, using the equivalence of 1- and 2-norms we have our desired result:

$$\|p\|_{\varepsilon^{-1}} + \|E\|_\varepsilon + \|H\|_\mu \leq \sqrt{3} \Big[\|p_0\|_{\varepsilon^{-1}} + \|E_0\|_\varepsilon + \|H_0\|_\mu + \|f_p\|_{L^1[0,T] \times L^2_{\varepsilon^{-1}}(\Omega)} + \|f_E\|_{L^1[0,T] \times L^2_{\varepsilon^{-1}}(\Omega)} + \|f_H\|_{L^1[0,T] \times L^2_{\mu^{-1}}(\Omega)} \Big].$$

Remark

As a result of above theorem, and by a standard argument, the solution to the variational formulation of the Maxwell's equations with given initial conditions has a unique solution.

Corollary (Energy Conservation)

If the forcing functions in the Maxwell's equations are all zero, that is, $f_p = 0$ and $f_E = f_H = 0$ and with initial conditions then:

$$\|p\|_{\varepsilon^{-1}}^2 + \|E\|_{\varepsilon}^2 + \|H\|_{\mu}^2 = \|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_{\varepsilon}^2 + \|H_0\|_{\mu}^2.$$

Spatial Discretization

Now we will show that how we can solve this problem computationally:

Spatial Discretization

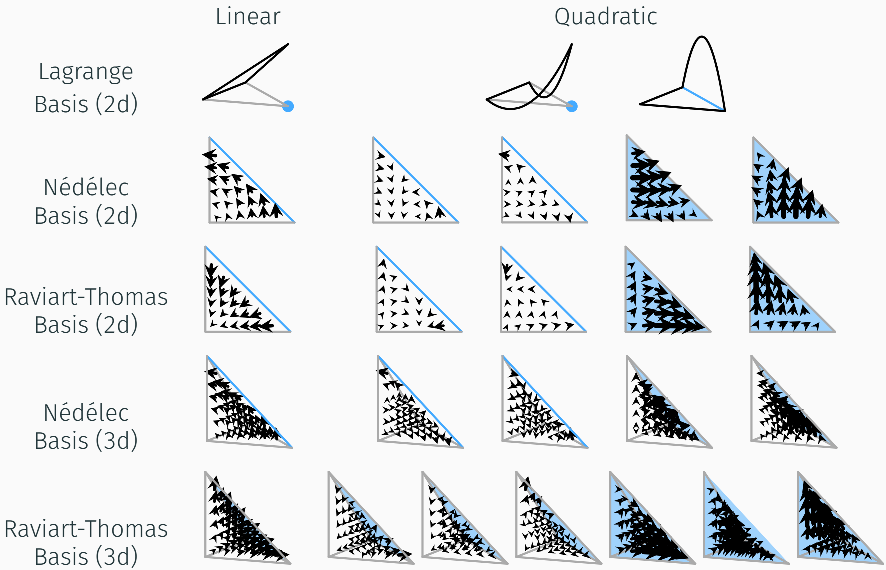
Now we will show that how we can solve this problem computationally:

$$\begin{array}{ccccccc}
\mathring{H}^1_{\varepsilon^{-1}}(\Omega) & \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} & \mathring{H}_\varepsilon(\operatorname{curl},\Omega) & \underset{\operatorname{curl}}{\overset{\operatorname{curl}}{\rightleftarrows}} & \mathring{H}_\mu(\operatorname{div},\Omega) & \underset{\operatorname{grad}}{\overset{-\operatorname{div}}{\rightleftarrows}} & L^2(\Omega) \\
\downarrow \Pi_h^0 & & \downarrow \Pi_h^1 & & \downarrow \Pi_h^2 & & \downarrow \Pi_h^3 \\
\mathring{H}^1_{\varepsilon^{-1}}(\Omega) \cap \mathcal{P}_r^-(\Omega) & \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} & \mathring{H}_\varepsilon(\operatorname{curl},\Omega) \cap \mathcal{P}_r^-(\Omega) & \underset{\operatorname{curl}}{\overset{\operatorname{curl}}{\rightleftarrows}} & \mathring{H}_\mu(\operatorname{div},\Omega) \cap \mathcal{P}_r^-(\Omega) & \underset{\operatorname{grad}}{\overset{-\operatorname{div}}{\rightleftarrows}} & L^2(\Omega) \cap \mathcal{P}_r^-(\Omega)
\end{array}$$

Here is a very small example that how we can use finite element method:

Strong form 2nd order PDE	$-\operatorname{div}\operatorname{grad} u = f$ on Ω
Dirichlet Boundary conditions	$u = 0$ on $\partial\Omega$
Weak form	$\langle \operatorname{grad} u, \operatorname{grad} v\rangle = \langle f, v\rangle$
Matrix form	$\left[S_{00}\right]\left[u\right] = \left[b\right]$
Function spaces	$u, v \in \mathring{H}^1_{\varepsilon^{-1}}(\Omega)$

Basis Elements



Time Discretizations

Consider uniform discretization of $[0, T]$ as $t^n := n\Delta t$, $n = 0, 1, \dots, N$ for some $\Delta t > 0$ being the fixed time step size such that $N\Delta t = T$.

Backward Euler:

$$\langle \frac{p^n - p^{n-1}}{\Delta t}, \tilde{p} \rangle - \langle \varepsilon E^n, \nabla \tilde{p} \rangle = \langle f_p^n, \tilde{p} \rangle, \tag{3a}$$

$$\langle \nabla p^n, \widetilde{E} \rangle + \langle \varepsilon \frac{E^n - E^{n-1}}{\Delta t}, \widetilde{E} \rangle - \langle H^n, \nabla \times \widetilde{E} \rangle = \langle f_E^n, \widetilde{E} \rangle, \tag{3b}$$

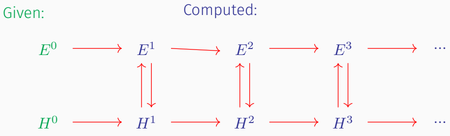
$$\langle \mu \frac{H^n - H^{n-1}}{\Delta t}, \widetilde{H} \rangle + \langle \nabla \times E^n, \widetilde{H} \rangle = \langle f_H^n, \widetilde{H} \rangle, \tag{3c}$$

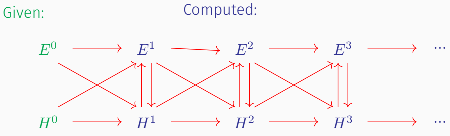
Crank Nicholson:

$$\langle \frac{p^n - p^{n-1}}{\Delta t}, \tilde{p} \rangle - \langle \frac{1}{2} \left(\varepsilon E^n + \varepsilon E^{n-1} \right), \nabla \tilde{p} \rangle = \langle \frac{1}{2} \left(f_p^n + f_p^{n-1} \right), \tilde{p} \rangle, \tag{4a}$$

$$\langle \frac{1}{2} \left(\nabla p^n + \nabla p^{n-1} \right), \widetilde{E} \rangle + \langle \varepsilon \frac{E^n - E^{n-1}}{\Delta t}, \widetilde{E} \rangle - \langle \frac{1}{2} \left(H^n + H^{n-1} \right), \nabla \times \widetilde{E} \rangle = \langle \frac{1}{2} \left(f_E^n + f_E^{n-1} \right), \widetilde{E} \rangle, \tag{4b}$$

$$\langle \mu \frac{H^n - H^{n-1}}{\Delta t}, \widetilde{H} \rangle + \langle \frac{1}{2} \left(\nabla \times E^n + \nabla \times E^{n-1} \right), \widetilde{H} \rangle = \langle \frac{1}{2} \left(f_H^n + f_H^{n-1} \right), \widetilde{H} \rangle, \tag{4c}$$

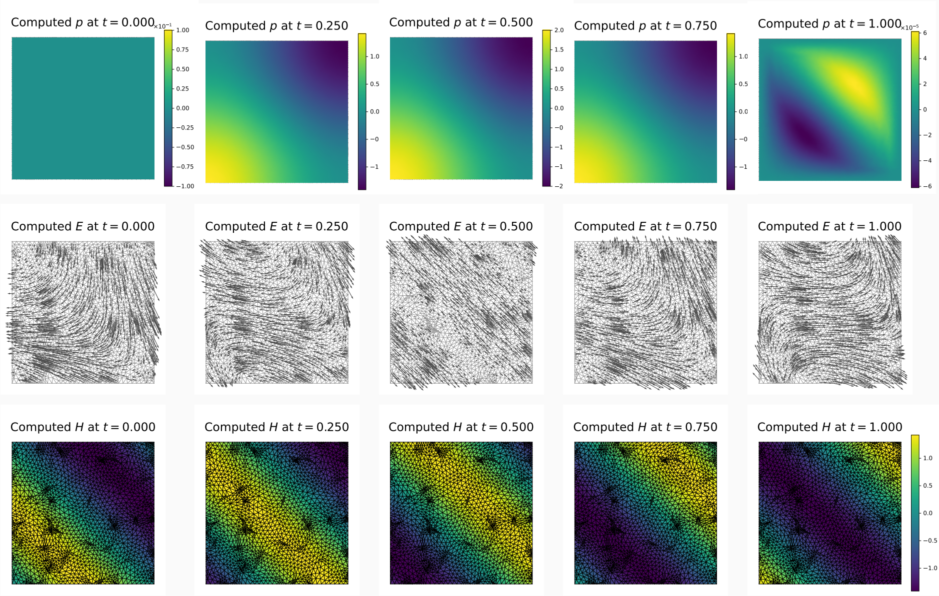




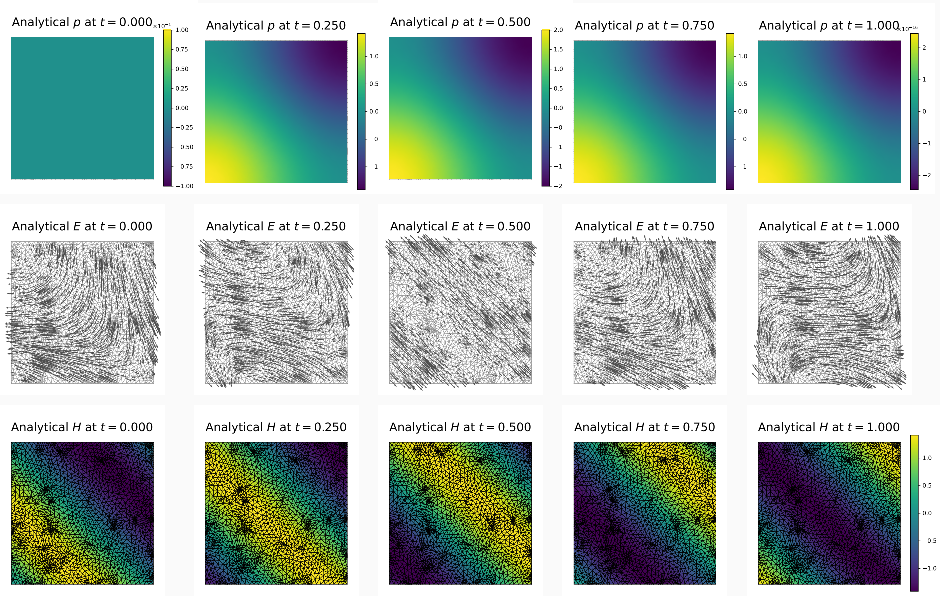
Python Code Fragment

```
# Crank Nicholson
# Setup the linear system for the solution of p, E and H
S_LHS = sprs.bmat([[1/dt*M00_g, -S01_g/2, None],
                   [-S01_g.T/2, -epsilon/dt*M11_g, mu*S12_g/2],
                   [None, S12_g.T/2, mu/dt*M22_g]], format='csr')
# Setup right hand side intermediate variables
bp_RHS = b_p/2 + 1/dt*M00_g*p[time_step - 1] + S01_g/2*E[time_step - 1]
bE_RHS = -b_E/2 + S01_g.T/2*p[time_step - 1] - epsilon/dt*M11_g*E[time_step - 1]
bH_RHS = b_H/2 - S12_g.T/2*E[time_step - 1] + mu/dt*M22_g*H[time_step - 1]

# Impose boundary conditions
S_LHS[boundary_vertex_indices] = 0
S_LHS[boundary_vertex_indices, boundary_vertex_indices] = 1
S_LHS[N0 + boundary_edge_indices] = 0
S_LHS[N0 + boundary_edge_indices, N0 + boundary_edge_indices] = 1
bp_RHS[boundary_vertex_indices] = p_bc
bE_RHS[boundary_edges] = E_boundary

# Setup the right hand side vector
b_RHS = np.concatenate((bp_RHS, bE_RHS, bH_RHS))

# Obtain the linear system solution for E and H
x = pEH_solver.solve(S_LHS, b_RHS)
```

2d Numerical Results on Unit Square (Crank-Nicholson)

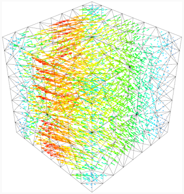


2d Analytical Results on Unit Square

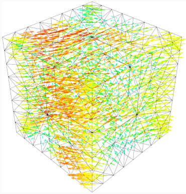


3d Numerical Results on Unit Cube (Crank-Nicholson)

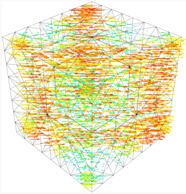
Quadratic E at $t = 0.000$



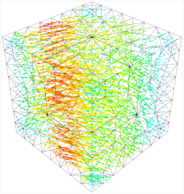
Quadratic E at $t = 0.250$



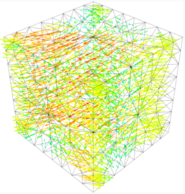
Quadratic E at $t = 0.500$



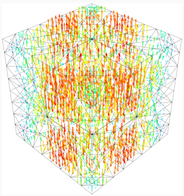
Quadratic E at $t = 0.750$



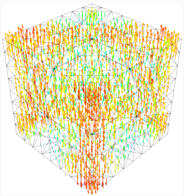
Quadratic E at $t = 1.000$



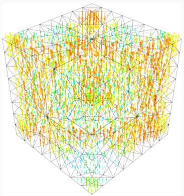
Quadratic H at $t = 0.000$



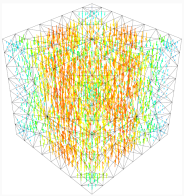
Quadratic H at $t = 0.250$



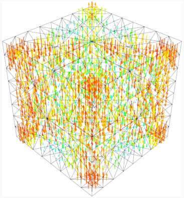
Quadratic H at $t = 0.500$



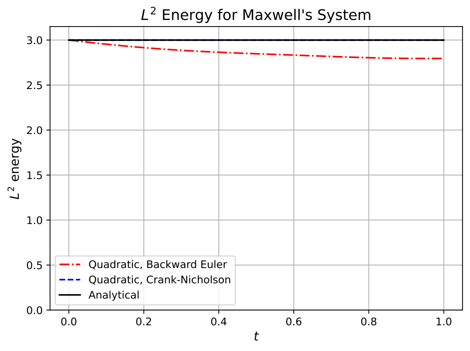
Quadratic H at $t = 0.750$

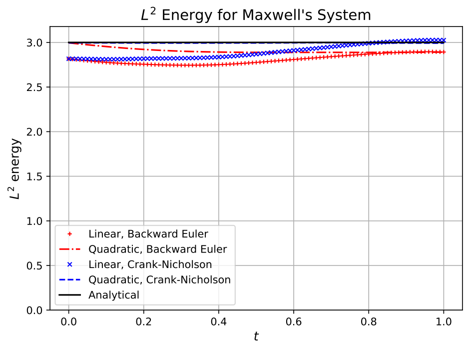


Quadratic H at $t = 1.000$



Numerical Results (Energy Plots)





Summary and Other Work

- Here are some more results that we can show for the mentioned Crank-Nicholson scheme for we have shown results:

Discrete Energy Estimate

$$\|p^N\|_{\varepsilon^{-1}} + \|E^N\|_{\varepsilon} + \|H^N\|_{\mu} \leq C.$$

Discrete Error Estimate

$$\|e_p^N\|_{\varepsilon^{-1}} + \|e_E^N\|_{\varepsilon} + \|e_H^N\|_{\mu} \leq C\left[(\Delta t)^2 + \|e_p^0\|_{\varepsilon^{-1}} + \|e_E^0\|_{\varepsilon} + \|e_H^0\|_{\mu}\right].$$

Full Error Estimate

$$\|e_{p_h}^N\|_{\varepsilon^{-1}} + \|e_{E_h}^N\|_{\varepsilon} + \|e_{H_h}^N\|_{\mu} \leq C\left[(\Delta t)^2 + h^r + h^r(\Delta t)^2\right].$$

- For more details, you can refer to our preprint: https://arxiv.org/abs/2310.20310

Thank You!

Questions?Remarks?Thoughts?

archanaa@iiitd.ac.in