The Correlation of Farey sequence

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(Based on joint work with Sneha Chaubey)

Farey Sequence

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Definition

Let Q be a positive integer and denote by $\mathcal{F}_Q$ the set of irreducible fractions in $[0, 1]$ whose denominator does not exceed Q,

$$\mathcal{F}_Q = \left\{ \frac{a}{q} : 0 \leq a \leq q \leq Q, (a, q) = 1 \right\}.$$

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Example

$$\mathcal{F}_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\}.$$

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- The cardinality of $\mathcal{F}_Q$

$$N(Q) = 1 + \sum_{q=1}^{Q} \phi(q) = \frac{3Q^2}{\pi^2} + O\left(Q \log Q\right).$$

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- If $\gamma_i = \frac{a_i}{q_i}$ is a fraction in $\mathcal{F}_Q$ and $\gamma_{i-1} = \frac{a_{i-1}}{q_{i-1}}$, $\gamma_{i+1} = \frac{a_{i+1}}{q_{i+1}}$ are adjacent fractions of γ_i then

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- The pairs of coprime integers (q, q') with $1 \leq q, q' \leq Q$, and $q + q' > Q$ are in one to one correspondence with the pairs of consecutive Farey fractions of order Q.

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- [Hardy and Littlewood, 1924] used Farey sequence in the circle method.
- [Ford, 1938] constructed the Ford circles using Farey fractions.

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- [Franel, 1924]

$$RH \iff \sum_{j=1}^{N(Q)} |\delta(j)| = O\left(Q^{1/2+\epsilon}\right)$$

where

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- [Landau, 1924]

$$RH \iff \sum_{j=1}^{N(Q)} \delta^2(j) = O\left(Q^{-1+\epsilon}\right).$$

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Let $\mathcal{F}$ be a finite set of cardinality N in $[0,1]$. The pair correlation measure $\mathcal{R}_{\mathcal{F}}(I)$ of a finite interval $I \subset \mathbb{R}$ is defined by

$$\frac{1}{N}\#\{(x,y) \in \mathcal{F}^2 : x \neq y,\ x - y \in \frac{1}{N}I + \mathbb{Z}\}.$$

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The limiting pair correlation measure of an increasing sequence $(\mathcal{F}_n)_n$, is given (if it exists) by

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If

$$\mathcal{R}(I) = \int_I g(x)dx,$$

then g is called the limiting pair correlation function of $(\mathcal{F}_n)_n$.

Montgomery's pair correlation conjecture

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[Montgomery, 1973] conjectured that, for any fixed $\alpha < \beta$,

$$N(\beta, T) := \sum_{\substack{0<\gamma,\gamma'\leq T \\ \frac{2\pi\alpha}{\log T}\leq\gamma-\gamma'\leq\frac{2\pi\beta}{\log T}}} 1 \sim \frac{T\log T}{2\pi}\int_{\alpha}^{\beta}\left(1-\left(\frac{Sin\pi u}{\pi u}\right)^2\right)du$$

$$+\frac{T\log T}{2\pi}\delta(\alpha,\beta),$$

where $\delta(\alpha, \beta) = 1$ if $0 \in [\alpha, \beta]$ and 0 otherwise.

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Theorem (Boca and Zaharescu, 2005)

The pair correlation function of $(\mathcal{F}_Q)_Q$ *is given by*

$$g(\lambda) = \frac{6}{\pi^2\lambda^2} \sum_{1 \le k < \frac{\pi^2\lambda}{3}} \phi(k) \log \frac{\pi^2\lambda}{3k}.$$

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Moreover, as $\lambda \to \infty$

$$g(\lambda) = 1 + O\left(\lambda^{-1}\right).$$

Visible lattice points along polynomials

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- For a fixed vector $(a_n, a_{n-1}, \cdots, a_1) \in \mathbb{Z}^n$ with $a_n \neq 0, a_i \geq 0$ for all i, and $\gcd(a_n, a_{n-1}, \cdots, a_1) = 1$, let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x$, we define

$$V := \left\{ (a,b) \in \mathbb{N}^2 \;\middle|\; \begin{array}{l} b = qP(a) \text{ for some } q \in \mathbb{Q}^+,\ \nexists\ (a',b') \in \mathbb{N}^2 \\ \text{such that } b' = q'P(a'),\ \text{and } a' < a,\ \ b' < b \end{array} \right\}$$

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- Denote

$$S := \{(a, b) \in \mathbb{N}^2 | \ \gcd(a_n a^n + a_{n-1} a^{n-1} + \cdots a_1 a, b) = 1\}$$

Generalized Farey fractions

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If $P(x) = x(x+1)$ then for instance

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The cardinality of $\mathcal{F}_{Q,P}$

$$\#\mathcal{F}_{Q,P} = \frac{Q^2}{2} \prod_p \left(1 - \frac{f_{a_n, a_{n-1}, \ldots, a_1}(p)}{p^2} \right) + O\left(Q^{1+\epsilon}\right),$$

where

$$f_{a_n, a_{n-1}, \ldots, a_1}(m) := |\{1 \leq d \leq m |\ a_n d^n + a_{n-1} d^{n-1} \ldots + a_1 d \equiv 0 \pmod{m}\}|.$$

Pair correlation of Generalized Farey fractions

Theorem (.B, Chaubey, 2023)

Let $c_1, c_2 \in \mathbb{Z}^+$ and $P(x) = c_2x^2 + c_1x$. The limiting pair correlation measure of the sequence $(\mathcal{F}_{Q,P})_Q$ under the GRH exists and is given by

$$\mathcal{S}(\Lambda) \ll \frac{(c_1c_2)^\epsilon}{\beta_p^{1+\epsilon}} \int_0^\Lambda \frac{1}{\lambda^{1-\epsilon}} \sum_{1 \le m < \frac{2\lambda}{\beta_p}} h_1(m) \log\left(\frac{2\lambda}{m\beta_p}\right) d\lambda,$$

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for any $\Lambda \ge 0$, where $\beta_p = \prod_p \left(1 - \frac{f_{c_2,c_1}(p)}{p^2}\right)$, and

$$h_1(m) = \frac{1}{m^\epsilon} \sum_{\substack{g_1|m \\ g_1|c_1}} \frac{1}{g_1} \sum_{\substack{g_2|\frac{m}{g_1} \\ g_2|c_1}} \frac{1}{g_2} \sum_{\delta|\frac{m}{g_1g_2}} \frac{1}{\delta}.$$

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Let $c_1, c_2, \cdots, c_\alpha \in \mathbb{Z}$ *with* $c_\alpha \neq 0$, $c_i \geq 0$ *and* $P(x) = x\mathcal{P}'(x)$, *where* $\mathcal{P}'(x) = c_\alpha x^{\alpha-1} + \cdots + c_2 x + 1$, $D = Disc(\mathcal{P}'(x))$. *The limiting pair correlation measure of the sequence* $(\mathcal{F}_{Q,P})_Q$ *under the GRH exists and is given by*

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for any $\Lambda \geq 0$, *where* $\beta_p = \prod_p \left(1 - \frac{f_{c_\alpha,\cdots,c_2,1}(p)}{p^2}\right)$, *and*

$$h_2(m) = \frac{1}{m^\epsilon} \sum_{\delta | m} \frac{1}{\delta}.$$

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Thank You!