Selection Mechanism in Non-Newtonian Saffman-Taylor Fingers

Mathematics Open House

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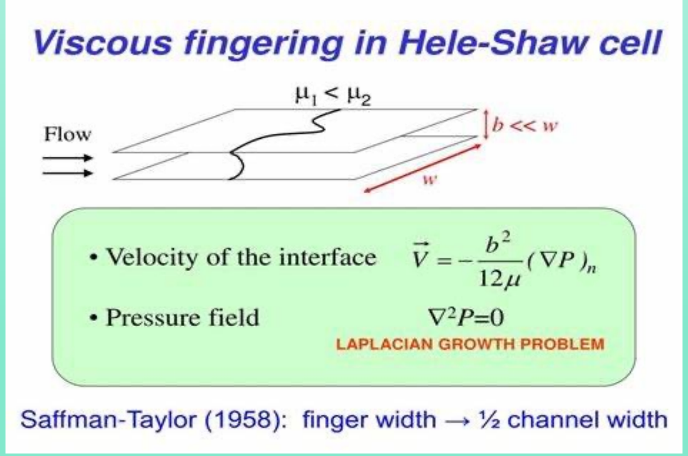
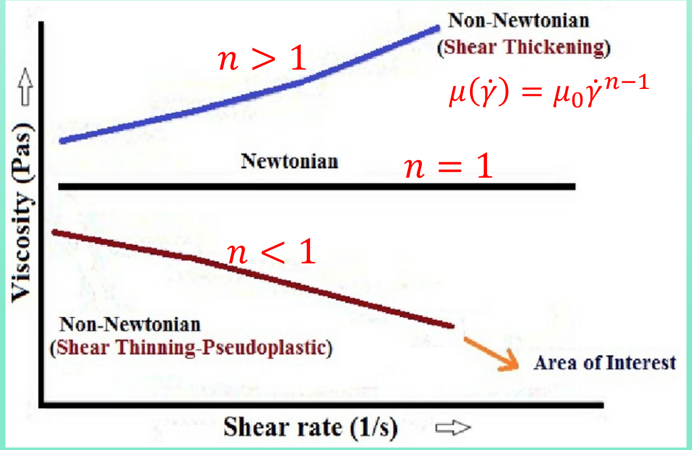
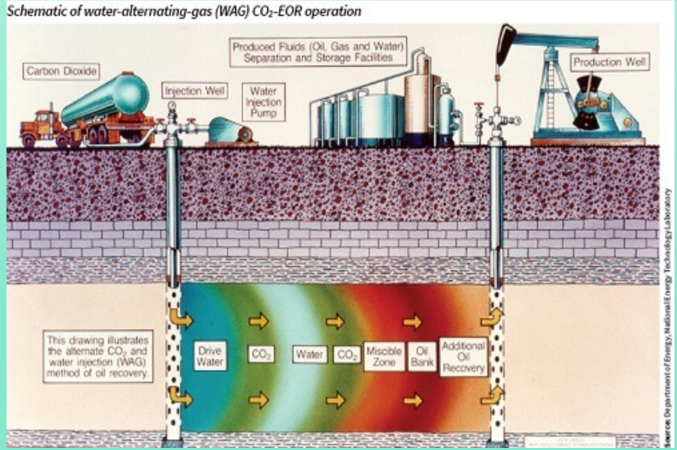


Viscous fingering



Newtonian Vs Non-Newtonian



Oil/Gas Recovery, CO2-EOR

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OUTLINE

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Mathematical Model

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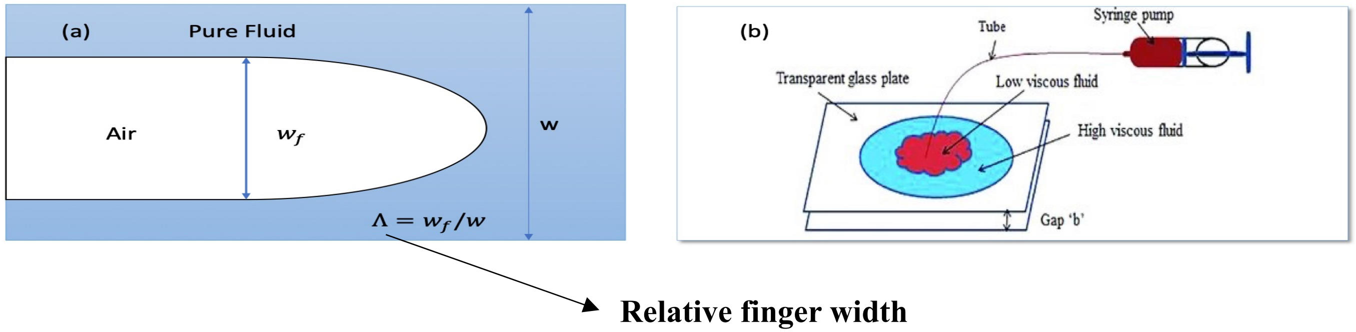
Introduction

Saffman-Taylor Instability (STI)

- Less viscous fluid pushes more viscous fluid through a narrow channel (a Hele-Shaw cell), forming finger like patterns at the interface referred to as viscous fingering phenomenon.



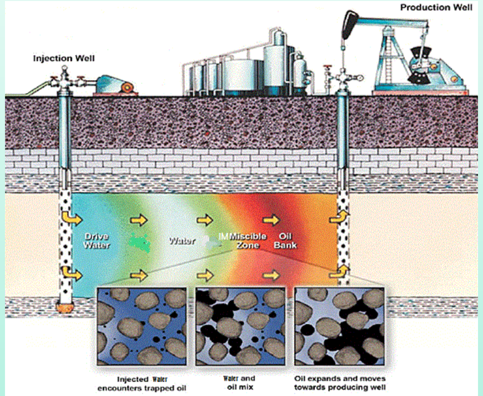
STI in a Hele-Shaw cell

- Problem first Studied by Saffman-Taylor at low Re (Math. and Phys. Sc. 245 (1958)) and analytical shape of the interface was given.

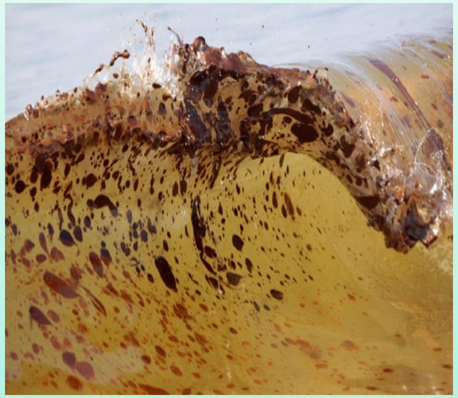
The finger-like complex structures are a **prototype for interfacial pattern generation**

Industrial Applications

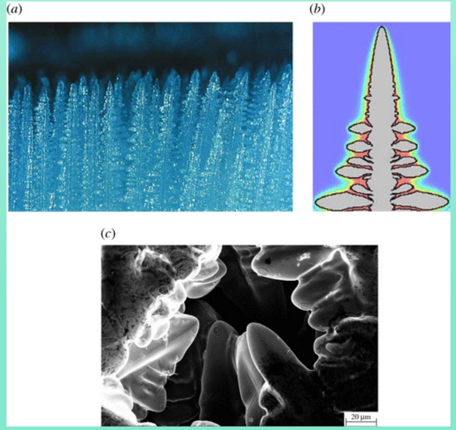
- Viscous fingering instabilities have received prolonged interest in theoretical and experimental studies as well as industry due to its vast practical applicability in
  - Recovery of oil/gas from the earth: Oil Industry
  - Cleaning up oil spill in ocean: Oil-Spill Clean-up
  - Hydrology and filtration
  - Dendrite/crystal formation
  - Fixed bed regeneration: Chemical processing
  - Fingering effect in porous media: Petroleum engineering literature



Oil Industry: Oil/Gas recovery, CO2-EOR



Oil spill cleanup



Dendrite & Crystal formation

Finger Selection Mechanism

- The relative finger width ($\lambda = \frac{Finger\ width\ (w)}{Channel\ Width\ (W)}$) is determined by the dimensionless Capillary parameter:

$$Ca = \frac{Viscous\ Force}{Capillary\ Force}$$

Viscous forces tend to narrow the finger

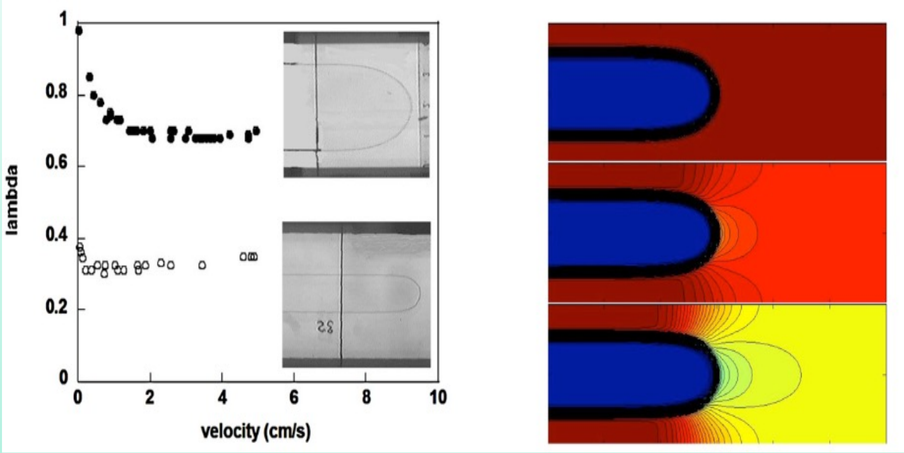
Capillary forces tend to widen the finger

- For large Ca, $\boldsymbol{\lambda}$ reaches a limiting (stable) value of about half the channel width ($\boldsymbol{\lambda = 0.5}$).

- The analytical expression for the shape of the finger (without surface tension) is given by Mc.Lean & Saffman in 1981:

$$x = \frac{w(1-\lambda)}{2\pi} \ln\left[\frac{1}{2}\left(1+\cos\left(\frac{2\pi y}{\lambda w}\right)\right)\right], 0 < \lambda < 0.5$$

But it fails to explain the the specific selection of the relative finger width $\boldsymbol{\lambda = 0.5}$.



Shear-Thinning and Shear-thickening phenomena

- A shear thinning or shear thickening polymeric liquid reveal a strong modification of the finger selection process.

- An analytical expression derived via a single, unified theory explaining both these features has remained elusive.

Analysis: Mathematical Model

Governing Equation & Model Formulation

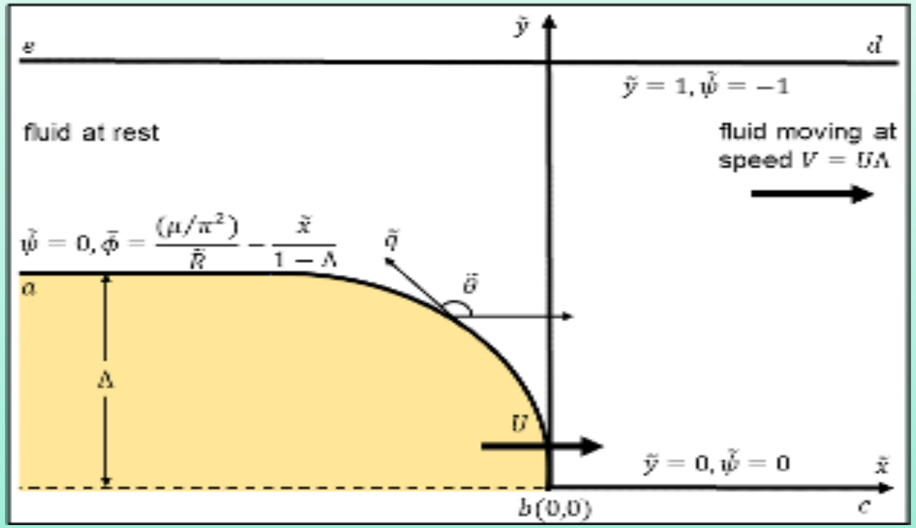
- Modified Darcy's Law: $\boldsymbol{u} = -\dfrac{b^2}{12\,\mu(\dot{\gamma})}\nabla p\,, \nabla.\boldsymbol{u} = 0$
- The viscosity of the non-Newtonian fluid:
  $$\mu(\dot{\gamma}) = \mu_0\dot{\gamma}^{n-1}$$
- Harmonicity of p gives: $p = -\dfrac{12\mu}{b^2}\phi$

  $\phi \longrightarrow$ Velocity potential
  $\psi \longrightarrow$ Stream Function (harmonic conjugate of ϕ)
  $\mathcal{F} = \phi + i\psi \longrightarrow$ Complex analytic function.

- Boundary conditions on ϕ and p are:

  $\left.\begin{aligned} \boldsymbol{n}.\boldsymbol{u} &= U\,sin\tilde{\theta} \\ p_0 - p &= \frac{T}{R} \end{aligned}\right\}$ on the advancing finger

  $u_y = 0$ on the walls: $y = \pm w$
  $u_x = U\lambda, u_y = 0$ as $x \to \infty, -w < y < w$
  $u_x = u_y = 0$ as $x \to -\infty, -w\lambda < |y| < w$



Advancing Finger

- WKB approximation:

  → This approximates the solution of the differential equation whose highest derivative is multiplied by a small parameter ε.

  → The solution is of the form of an asymptotic series expansion.

Integro-Differential Equation

$$\nu_0 \left\{ \frac{1}{b(1-\Lambda)} \right\}^{(1-n)} qs \frac{d}{ds} \left(q^{(2-n)} s \frac{d\theta}{ds} \right) = q - \cos\theta,$$

$$\log q = -\frac{s}{\pi} \mathcal{P} \int_0^1 \frac{\theta(s')}{s'(s'-s)} ds', \qquad s \in [0,\ 1],$$

The Boundary Conditions:

$$\theta(0) = 0, \quad q(0) = 1, \quad \theta(1) = -\pi/2, \quad q(1) = 0,$$

Where, $\nu_0 = \dfrac{Tb^2\pi^2}{12\mu_0 U w^2 (1-\Lambda)^2}.$

Results

- The leading order term (zero surface tension) $\boldsymbol{\nu_0 = 0}$ gives an explicit solution but fails to determine a **unique** value of λ.
- Our main result is the following proposition, providing us with a unique solution for the system of equations in the limit $\nu_0 \to 0$:
- **Proposition:** In the limit $\nu_0 \to 0$, (q,θ) satisfying the equations along with the boundary conditions has a unique solution provided λ satisfies the following relation (accurate unto the leading order in ν_0),

$$\lambda \sim \begin{cases} \frac{1}{2} - \mathcal{O}\left(\nu_0^{-\frac{1}{2}}\right), & n < 1, \lambda < \frac{1}{2}, \quad \textit{for shear thinning} \\ \frac{1}{2} + \mathcal{O}\left(\nu_0^{\frac{2}{4-n}}\right), & n \geq 1, \lambda > \frac{1}{2}, \quad \textit{for shear thickening} \end{cases}$$

- **Solvability Theory:** Consider $\Theta \in C^{\infty}(\mathbb{R})$ and a differential operator $\mathcal{L}$ such that,

$$\mathcal{L}\Theta = \bar{R}$$

Where, $\bar{R} \in L^1(\mathbb{R})$. Let Θ_0 be the null eigenvector of the adjoint of $\mathcal{L}$, or $\Theta_0 \in N(\mathcal{L}^{\dagger})$. Further define the cusp function, $C \in L^1(\mathbb{R})$, such that

$$C = \int_{-\infty}^{\infty} d\eta \Theta_0 \bar{R}(\eta)$$

If Θ exists uniquely, then $\boldsymbol{C \equiv 0}$ (E. Corvera 1995).

- A good agreement is found near the power-law exponent, n = 1 (Hong & Langer 1986).

Source: Ghosh D. et al., *SIAM J. on Appl. Math.*, *83*(2), 329-353(2023)

Numerical and Experimental Validation

DISPERSION RELATION

The dispersion relation in a rectilinear channel for the power law fluids can be expressed as:

$$\omega = iU[\alpha - B\alpha^3 w^2]$$

$$B(\dot{\gamma}) = \frac{Tb^2}{12\mu(\dot{\gamma})Uw^2}$$: Control parameter.

Concludes

- Existence of the most unstable temporal mode obtained by setting

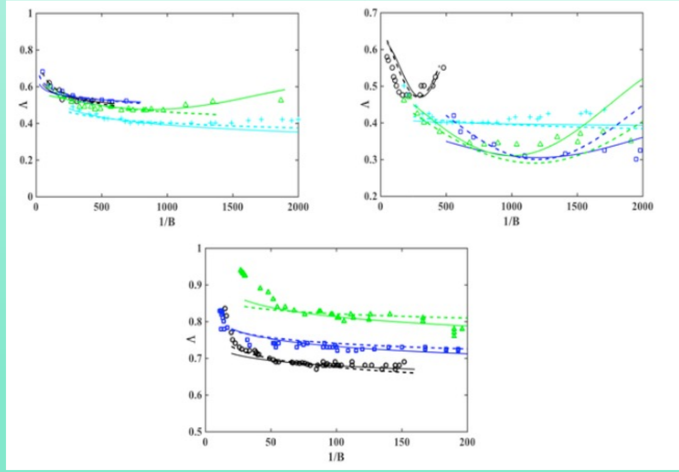
$$\frac{d\omega}{d\alpha} = 0 \Rightarrow \alpha_{temp} = (\omega\sqrt{3B})^{-1}$$

- The relative finger width of the advancing interface,

$$\lambda_{temp} = \frac{2\pi}{\alpha_{temp}}$$

EXPERIMENTAL OBSERVATION

A reasonably good agreement is found between the experimental data, the one computed with the linearized model (i.e. λ_{temp}) and the theoretical estimates of our results, for large values of 1/B.



Finger width as a function of 1/B for solutions of (a) Xanthane for concentrations of 50 ppm (O) 100 ppm (□), 500 ppm (△), 1000 ppm (+), and at fixed cell geometry, w = 2.0cm and b = 0.25mm , (b) Xanthane at different cell geometries, w = 2.0cm and b = 0.66mm (O) , w = 2.0cm and b = 0.25mm ( ), w = 4.0cm and b = 0.5mm (△), w = 4.0cm and b = 0.25mm (□), and at fixed concentration of 1000 ppm, (c) PEO for concentrations of 5 ppm (O) , 50 ppm (□), 500 ppm (△), and at fixed cell geometry, w = 2.0cm and b = 0.5mm. (----) predictions from the linear stability analysis. (——) predictions from the theoretical estimate.

Conclusion

- An analytical treatment of predicting the Saffman-Taylor fingers for a class of non-Newtonian fluids is provided.
- A systematic description is provided here for the singular perturbation introduced by the viscous and the capillary forces leads to a solvability mechanism for pattern selection.
- Our results extend the classical results for Newtonian fluids of Hong and Langer (1986), Shraiman (1986) and Combescot et al. (1986) ⟹ in the limit of small ν_0, $\lambda - \frac{1}{2} \sim \nu_0^{2/3}$.
- A future endeavour is the perturbative expansion of the model via the radius of the fingertip to explain the selection mechanism for **strong shear thinning fluids**.
- In our analysis, WKB expansion and conformal mapping technique had been used. So, the next aim is to derive similar result using other conformal mapping technique, such as **generalised Polubarinova-Galin equation**.

Publications/Preprints

- D. Bansal, **D. Ghosh**, S. Sircar; Selection mechanism in non-Newtonian Saffman-Taylor fingers. ***SIAM*** Journal on Applied Mathematics; 83(2): 329-353 (2023). https://doi.org/10.1137/22M1485838
- **D. Ghosh**, T. Chauhan, S. Sircar; Implicit-explicit time integration method for fractional advection-reaction-diffusion equations (under review (2023)). https://doi.org/10.48550/arXiv.2301.06507
- D. Bansal, **D. Ghosh**, S. Sircar; Spatiotemporal linear stability of viscoelastic free shear flows: Nonaffine response regime. ***Physics of Fluids***; 33(5) : 054106 (2021). https://doi.org/10.1063/5.0049504

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THANK YOU

QUESTIONS?

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