Constructing New Diffeomorphisms with Non-Ergodic Generic Measures

Divya Khurana

Department of Mathematics, IIIT-D

MATHEMATICS OPEN HOUSE

Outline

A little about dynamical system

Let (X, T) be a dynamical system, where X is a metric space and $T : X \longrightarrow X$ a map.

Interest: The long time behaviour of the orbits $T^n x$, as $n \to \infty$.

- How does the orbit of a point behave?
- Does it distribute nicely over X?
- Does it tend to cluster around certain points?

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example

$T : \mathbb{T}^1 \to \mathbb{T}^1$ defined by $T(x) = x + \alpha \mod 1$.
If α rational $\implies \{f^n x\}$ is periodic.
If α irrational $\implies \{T^n x\}$ is dense over $\mathbb{T}^1$.

Different mixing properties

- Let $(X, \mathcal{B}, \mu)$ be a measure space. $(X, \mathcal{B}, \mu, T)$ is called a measure-preserving transformation(mpt), if $T : X \longrightarrow X$ is measurable, i.e. $A \in \mathcal{B} \implies T^{-1}(A) \in \mathcal{B}$, and measure preserving, i.e. $\mu(T^{-1}(A)) = \mu(A) \ \forall \ A \in \mathcal{B}$.
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- Ergodic Theory is the branch of Dynamical Systems studying properties of the iterates of mpt.
- A mpt $(X, \mathcal{B}, \mu, T)$ is called **ergodic**, if every invariant set A satisfies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Different mixing properties

- A mpt $(X, \mathcal{B}, \mu, T)$ is said to be **strong mixing**, if for $A, B \in \mathcal{B}$,

$$|\mu(A \cap T^{-n}B) - \mu(A)\mu(B)| \to 0 \text{ as } n \to \infty$$

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- Strong Mixing $\implies$ Weak Mixing $\implies$ Ergodicity.

Some definition

Definition (Generic point)

A point $x \in X$ is a *generic point* for $(X, \mathcal{B}, \mu, T)$, if for every continuous function $\phi : X \longrightarrow \mathbb{R}$, we have

$$\frac{1}{N}\sum_{i=0}^{N-1}\phi(T^i x) \longrightarrow \int_X \phi d\mu.$$

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A measure μ is called **generic measure** if it has a generic point.

- Generic points form a powerful tool in ergodic theory in quantifying the difference between two invariant measures.
- From the Ergodic Theorem: For an ergodic measure almost every point is a generic point.

A Motivation: Literature Review

- But non-ergodic measures may or may not have generic points.
- Gelfert and Kwietniak [2014] provided an example of the subshift space with exactly two ergodic measures such that every non-ergodic measure fails to have a generic point.
- Chaika and Masur [2015] constructed an example of nonergodic generic measure with two ergodic measures on 6-interval exchange transformation.

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- To the best of our knowledge, no differentiable or even continuous map exists on any manifold.
- Hence, the question arises for a setting like a smooth dynamical system: Is it possible to have volume as a non-ergodic generic measure?

Preview of Results

The analog of Chaika and Masur's result for smooth diffeomorphism.

Theorem (K.)

For any $r \in \mathbb{N}$, there exists a minimal smooth diffeomorphism $T \in \mathit{Diff}^\infty(\mathbb{T}^2, \mu)$ that has exactly r ergodic invariant measures(they are absolutely continuous wrt Lebesgue), and the Lebesgue measure is generic wrt to this diffeomorphism.

Upgrade: To Topological and a Mixing Property

Question

With currently known technique:

- Can we upgrade the same construction for weak mixing and have generic non-ergodic measure?

History

D.V. Anosov by K. Jacobs, CC BY-SA 2.0 de.

A. Katok, CC BY-SA 3.0.

- The Anosov Katok method is a technique for constructing examples of dynamical systems satisfying "interesting" properties.
- It was invented by D.V. Anosov and A. Katok in 1970.
- Also known as the "Approximation by Conjugation" method or the "AbC" method.

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- It was invented by D.V. Anosov and A. Katok in 1970.
- Also known as the "Approximation by Conjugation" method or the "AbC" method.
- This method has been very successful in constructing examples of smooth dynamical systems on smooth compact manifolds admitting a nontrivial action $\{S_t\}_t \in \mathbb{T}^1$ of the circle $\mathbb{T}^1$.

A technique: AbC Method(overview)

- Let's denote S_t, a measure preserving circle $\mathbb{T}^1$ action on $\mathbb{T}^2 = \mathbb{R}^2 \backslash \mathbb{Z}^2$:
$$S_t(x_1, x_2) = (x_1 + t, x_2).$$
- The method involves the construction of a required diffeomorphism T as the limit of periodic diffeomorphisms
$$T = \lim_{n \longrightarrow \infty} T_n$$
- The limit is taken in the appropriate topology.

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$$T = \lim_{n \longrightarrow \infty} T_n$$
- The limit is taken in the appropriate topology.
- The sequence of T_n are defined iteratively as
$$T_n = H_n \circ S_{\alpha_n} \circ H_n^{-1}$$
where $\alpha_n = \frac{p_n}{q_n} \in \mathbb{Q}$ and $H_n = h_1 \ldots h_n$ where h_n is a measure preserving diffeomorphism satisfying $S_{\alpha_{n-1}} \circ h_n = h_n \circ S_{\alpha_{n-1}}$.

The Magic trick

At the n+1-th step,
We have α_n, H_n and T_n. Construct iteratively a measure preserving diffeomophism h_{n+1} and the rational number α_{n+1}.
We first construct diffeomorphism h_{n+1} such that
h_{n+1} **commutes with** S_{α_n} :

$$T_n = H_n \circ (h_{n+1} \circ h_{n+1}^{-1}) \circ S_{\alpha_n} \circ H_n^{-1}$$

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$$T_{n+1} = H_n \circ (h_{n+1} \circ S_{\alpha_{n+1}} \circ h_{n+1}^{-1}) \circ H_n^{-1}$$

Then, subsequent T_{n+1} can be made close enough to T_n in the appropriate topology.

Key Idea

1. Choice of rationals α_n's allows the convergence of the sequence T_n in the required topology.
2. At each stage n, conjugation by h_n perturbes the circle orbit such that it distributes the orbit through the space in the required sense.

Key Idea

1. Choice of rationals α_n's allows the convergence of the sequence T_n in the required topology.
2. At each stage n, conjugation by h_n perturbes the circle orbit such that it distributes the orbit through the space in the required sense.
3. The asymptotic version of that property is transferred into the final map T.

Some applications

- Fayad and Saprykina [2005] produced an example of a smooth weak mixing diffeomorphism on $\mathbb{T}^2, \mathbb{D}^2$ and $\mathbb{A}$ for any Liouvillian number.
- Windsor A. [2001] constructed an example of a minimal smooth diffeomorphism with exactly r ergodic invariant measures on $\mathbb{T}^2$.

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- Windsor A. [2001] constructed an example of a minimal smooth diffeomorphism with exactly r ergodic invariant measures on $\mathbb{T}^2$.
- Banerjee S. and Kunde P. [2019] constructed an example of a minimal real analytic diffeomorphsim with exactly r ergodic invariant measures on $\mathbb{T}^2$

Main Result

Theorem (K.)

For any natural number r, there exists a minimal $T \in \text{Diff}^\infty(\mathbb{T}^2, \mu)$ such that the Lebesgue measure is a generic measure for T and exactly r invariant measures, $\mu_1, \mu_2, \ldots, \mu_r$, such that T is weakly mixing w.r.t. each of these measures.

Generalize: To a large set of Generic points

Question

- Can we construct more generic points for a non-ergodic map?
- Can we estimate the size of a set in terms of Hausdorff Dimension?

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Theorem (K.)

There exist a smooth diffeomorphism $T \in Diff^{\infty}(\mathbb{T}^2, \mu)$ constructed by the approximation of conjugation method, such that the set A containing all the generic points of T has

$$\log_3 2 \leq dim_H(A) \leq 1 + \log_3 2$$

and $\mu(A) = 0$.

Generalize: To non-generic points

Question

Can we estimate the set of **non-generic** points for the case of ergodic measure ?

Generalize: To non-generic points

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Can we estimate the set of **non-generic** points for the case of ergodic measure ?

Theorem (K.)

For any $1 < \beta < 2$, there exists a smooth ergodic diffeomorphism $T \in \mathit{Diff}^{\infty}(\mathbb{T}^2, \mu)$ constructed by the approximation of the conjugation method, such that B_β consists of all the **non-generic points** *of T has*

$$\beta - 1 \leq \mathit{dim}_H(B_\beta) \leq \beta,$$

and $\mu(B_\beta) = 0$

A Basic idea

Goal: Construct $T \in \text{Diff}^{\infty}(\mathbb{T}^2, \mu)$ such that the set containing all the generic points has a non-trivial Hausdorff dimension.

Set up: Introduce two parts of the torus with distinct aims.

- A Generic zone
- A Non-Generic Zone
- Partition the above space into more refined elements explicitly.

A Basic idea

Goal: Construct $T \in \text{Diff}^\infty(\mathbb{T}^2, \mu)$ such that the set containing all the generic points has a non-trivial Hausdorff dimension.

Set up: Introduce two parts of the torus with distinct aims.

- A Generic zone
- A Non-Generic Zone
- Partition the above space into more refined elements explicitly.
- Define the map h_n suitable for combinatorics of T_n.
- Show the convergence $T_n \to T \in \text{Diff}^\infty(\mathbb{T}^2, \mu)$ and the limit has that targeted property.

Sketch Idea: Theorem

A set of generic points with positive Hausdorff dimension

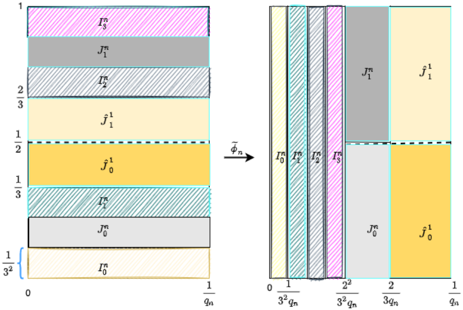

Figure: An example of the action of h_{n+1} where $k_n = 3, q_n = 3$

- Partition the vertical axis $\mathbb{T}^1$ with respect to the middle third cantor set C.

Sketch Idea: Theorem

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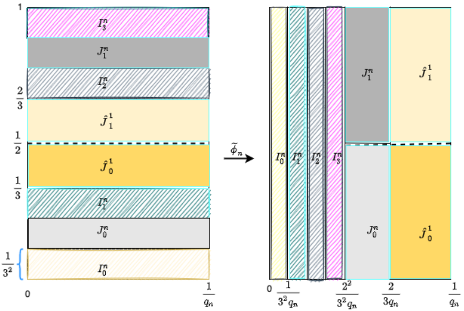
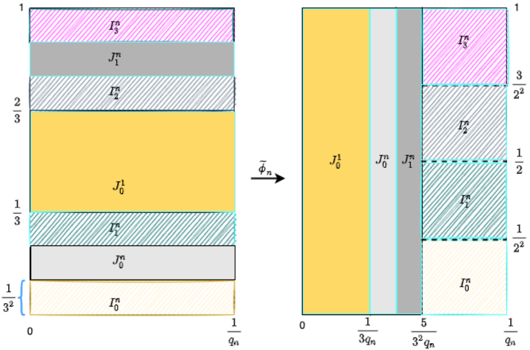


Figure: An example of the action of h_{n+1} where $k_n = 3, q_n = 3$

- Partition the vertical axis $\mathbb{T}^1$ with respect to the middle third cantor set C.
- Choose $A = \mathbb{T}^1 \times C$ as generic zone.
- Choose $B = \mathbb{T}^1 \times C^c$ as non-generic zone, where C^c is complement of C

Sketch Idea: Theorem

A set of non-generic points with positive Hausdorff dimension



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References

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Thank you...

Rough