Logics of Formal Inconsistency

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Contradictions

Contradictions are important.
In fact, in many cases, they are **quite informative!**
Suppose if we ask a yes-no question to two people: "Does the person A live at place B?"
Exactly one of the three following distinct scenarios is possible:

1. They might both say 'yes'.
2. They might both say 'no'.
3. One of them might say 'yes' while the other says 'no'.

In no situation we can be sure whether A lives at B or not , but only in the last scenario where a contradiction appears, we are sure to have received wrong information from one of the sources.

Some Rules and Definitions

Given a set *For* of formulas,
a consequence relation over For, $\vdash \subseteq \mathbf{P}(For) \times For$
and a logic **L**, $\mathbf{L} =< For, \vdash >$
Any set $\Gamma \subseteq For$ is called a theory of **L**.
Let Γ be a theory of **L**.
Γ is contradictory if:
$\exists\alpha(\Gamma \vdash \alpha \text{ and } \Gamma \vdash \neg\alpha)$
A theory Γ is trivial if :
$\forall\alpha(\Gamma \vdash \alpha)$
A theory Γ is explosive if:
$\forall\alpha\forall\beta(\Gamma, \alpha, \neg\alpha \vdash \beta)$

L is trivial if all of its theories are trivial.
L is contradictory if all of its theories are contradictory.
L is explosive if all of its theories are explosive.

(Reflexivity)
$\alpha \in \Gamma$ implies $\Gamma \vdash \alpha$.

(Monotonicity)
$(\Delta \vdash \alpha$ *and* $\Delta \subseteq \Gamma)$ implies $\Gamma \vdash \alpha$.

(Transitivity/Cut for Sets)
For all $\alpha \in \Delta$, $\Gamma \vdash \alpha$ and $\Delta \vdash \beta$ implies $\Gamma \vdash \beta$.

Paraconsistent Logic

Principle of Explosion (**L** is explosive)
$\forall\Gamma\forall\alpha\forall\beta(\Gamma,\ \alpha,\ \neg\alpha \vdash \beta)$

A logic is **paraconsistent** if Principle of Explosion fails,i.e.,
$\exists\Gamma\exists\alpha\exists\beta(\Gamma,\ \alpha,\ \neg\alpha \nvdash \beta)$

Application of Paraconsistent Logic

Suppose we have constructed a computer system to manage traffic lights. Because it is important to know that whether traffic lights are functioning properly, we might attach a sensor to the traffic lights to determine whether it is functioning properly or not.

But sensors eventually break so we might want to add a second sensor as a backup to the first.

Now if one of the sensors does break, it seems that our system may well receive a message from one sensor, saying that the light is functioning and from the other receive message saying that light is not functioning.

Now if we have programmed our computer to use classical logic, we are in trouble at this point. But if we have written our code using paraconsistent logic, we are safe.

Gentle Principle of Explosion

Let $\bigcirc(p)$ be a set of formulas (possibly empty) which depends only on the propositional variable p, satisfying the following:
There are formulas α and β such that
$\bigcirc(\alpha),\ \alpha \nvdash \beta$;
$\bigcirc(\alpha),\ \neg\alpha \nvdash \beta$.
(Here , $\bigcirc(\phi) = \{\psi(\phi) : \psi(p) \in \bigcirc(p)\}$)
Then:
A theory Γ is **gently explosive** (with respect to $\bigcirc(p)$) if:
$\forall\alpha\ \forall\beta \quad (\Gamma, \bigcirc(\alpha), \alpha, \neg\alpha \vdash \beta)$
A logic **L** is said to be **gently explosive** when there is a set $\bigcirc(p)$ such that all the theories of **L** are gently explosive (with respect to $\bigcirc(p)$).

Definition

A Logic of Formal Inconsistency (**LFI**) is any **gently explosive paraconsistent logic**, that is, any logic in which explosion does **not** hold good while gentle explosion holds good.
In other words, a logic **L** is an **LFI** (with respect to a negation $\neg$) if:

1. $\exists\, \Gamma\, \exists \alpha\, \exists \beta\, (\Gamma, \alpha, \neg\alpha \nvdash \beta)$ and
2. there exists a set of formulas $\bigcirc(p)$ such that $\forall \Gamma\, \forall \alpha\, \forall \beta\, (\Gamma, \bigcirc(\alpha), \alpha, \neg\alpha \vdash \beta)$

mbC(a basic LFI)

1. **'bC'** stands for basic property of consistency.
2. It is an **extension of classical positive propositional logic**.
3. It has **sound and complete** bivalued semantics.
4. $\circ A$ and $\neg(A \wedge \neg A)$ are not equivalent where A is some formula i.e. **consistency and non-contradiction are not equivalent** in mbC.

Syntax

Let L_1 be a language with a denumerable set of sentence symbols $\{p_1, p_2, p_3 \ldots\}$, the set of connectives $\{\circ, \neg, \wedge, \vee, \rightarrow\}$ and parentheses. The set of formulas of L_1 is obtained recursively in the usual way. The logic **mbC** is defined over the language L_1 by the following axiom schemas:

Axiom 1 $A \rightarrow (B \rightarrow A)$

Axiom 2 $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

Axiom 3 $A \rightarrow (B \rightarrow (A \wedge B))$

Axiom 4 (i) $(A \wedge B) \rightarrow A$ (ii) $(A \wedge B) \rightarrow B$

Axiom 5 (i) $A \rightarrow (A \vee B)$ (ii) $B \rightarrow (A \vee B)$

Axiom 6 $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$

Axiom 7 $A \vee (A \rightarrow B)$

Axiom 8 $A \vee \neg A$

Axiom**(bc1)** $\circ A \rightarrow (A \rightarrow (\neg A \rightarrow B))$

...Continued

Inference Rule: *Modus Ponens*

Positive Classical propositional logic, $\mathbf{CPL}^+$ is given by Axioms 1-7 plus Modus ponens.

Due to Axiom (bc1), **mbC is gently explosive**.

For some A and B:

$A, \neg A \nvdash B$,

$\circ A, A \nvdash B$,

$\circ A, \neg A \nvdash B$

While for every A and B :

$\circ A, A, \neg A \vdash B$

Semantics for mbC

1. An **mbC** valuation $v : For \rightarrow \{0, 1\}$ (over L_1) satisfies the following:
   2.1 $v(A \wedge B) = 1$ iff $v(A) = 1$ and $v(B) = 1$.
   2.2 $v(A \vee B) = 1$ iff $v(A) = 1$ or $v(B) = 1$.
   2.3 $v(A \rightarrow B) = 1$ iff $v(A) = 0$ or $v(B) = 1$.
   2.4 $v(\neg A) = 0$ implies $v(A) = 1$.
   2.5 $v(\circ A) = 1$ implies $v(A) = 0$ or $v(\neg A) = 0$
2. Logical consequence is defined as follows:
   $\Gamma \vDash_{mbC} A$ iff for every valuation v, for all $B \in \Gamma$, $v(B) = 1$ implies $v(A) = 1$.

Here, we can think of 0 and 1 not as false and true (respt.) but rather as absence and presence of evidence.

1. $v(A) = 1$ means 'there is evidence that A is true'.
2. $v(A) = 0$ means 'there is no evidence that A is true'.
3. $v(\neg A) = 1$ means 'there is evidence that A is false'.
4. $v(\neg A) = 0$ means 'there is no evidence that A is false'.

Suppose, in particular, we take the case
$v(A) = 1$, $v(\neg A) = 0$ or (vice versa) and $v(\circ A) = 0$.
Then $v(\neg(A \wedge \neg A)) = 1$, hence $\circ A$ and $\neg(A \wedge \neg A)$ are not equivalent.
mbC is sound and complete.

THANK YOU