Adaptive Finite Element Method

Bridging the Gap in Computational Modeling

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Talk Overview

- Finite Element Method for 1d and 2d Poisson's Equation. Physical significance and characteristics of Poisson's Equation.
- Understand the weak formulation via 1d Poisson's Equation by considering a Test function in a finite dimensional space.
- Examples of Basis functions.
- A problem solved via FEM.
- A prior and a posteriori error estimates for 2d Poisson's Equation.
- What is Adaptive Finite Element Method?
- Example of an a posteriori error estimate for AFEM.

- Poisson's equation is a partial differential equation (PDE) that helps model physical phenomena like heat conduction, electrostatic fields, and fluid flow.
- A numerical method to solve such a PDE is the finite element method (FEM).
- In FEM, we compute a Galerkin or weak formulation of the PDE by taking an inner product and seeking equality for arbitrary choices of functions.
- FEM becomes computational by selecting the spaces for choosing these functions to be finite dimensional ones.
    - A basis is constructed locally using a discretization of the problem domain
    - The local basis are glued together with some continuity conditions to ensure a piecewise approximation of the solution

$$-u''(x) = f(x), \quad -1 < x < 1, \quad u(-1) = u(1) = 0.$$

- Standard Weak Formulation

$$\int_{-1}^{1} -u''(x)v(x)dx = \int_{-1}^{1} f(x)v(x)dx, \quad \text{for all } v \in V$$

where V is space of functions in $L^2(-1,1)$ whose first derivative is also in $L^2(-1,1)$ and $v(-1) = 0 = v(1)$.

$$\implies \int_{-1}^{1} u'(x)v'(x)dx - \cancel{\Big[u'(x)v(x)\Big]_{-1}^{1}} = \int_{-1}^{1} f(x)v(x)dx$$

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- Weak formulation:

$$\int_{-1}^{1} u'(x)v'(x)dx = \int_{-1}^{1} f(x)v(x)dx, \quad \text{for all } v \in V$$

  - Finite – finite subspace V_h of V
  - Element – V_h basis compactly supported.
  - Galerkin's method – seek $u \in V_h$ as well.
- Need finite $V_h \subset V$:
  - Divide $(-1, 1)$ into subintervals,
  - Pick $\{\phi_i\}$ with compact support; $\phi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

$$\implies \sum_{j,\,i} \int_{-1}^{1} u_j \, \phi_j'(x) \, \phi_i'(x) v_i dx = \sum_{i} \int_{-1}^{1} f(x)\phi_i(x)v_i \, dx$$

$$\implies \sum_{j,\,i} \int_{-1}^{1} u_j\phi_j'(x)\phi_i'(x)v_i dx = \sum_{k,\,i} \int_{-1}^{1} f_k\phi_k(x)\phi_i(x)v_i dx$$

- Resulting finite dimensional problem: $Ku = Mf$
- $\{\phi_i\}$ usually piecewise polynomials; simplest is linears.

Finite Element Method for 2d Poisson's Equation

$$-\operatorname{div}\operatorname{grad} u = f \quad \text{on} \quad \Omega \subseteq \mathbb{R}^2 \qquad \text{and} \qquad u = 0 \text{ on } \partial\Omega.$$

- If $f \in L^2(\Omega)$, then $u \in H^2(\Omega)$ in the strong form of PDE
- Weak Formulation: Using inner product and integration by parts we seek $u \in \mathring{H}^1(\Omega)$ such that:

$$\langle \text{grad } u, \text{grad } v\rangle - \langle \text{grad } u \cdot n, v\rangle_{\partial\Omega} = \langle f, v\rangle$$

  for all $v \in \mathring{H}^1(\Omega)$ and where n is the unit outward normal at the boundary $\partial\Omega$.
- Boundary conditions: The boundary conditions may be nonhomogeneous ($\neq 0$) or homogeneous ($= 0$), and can be of the following different kinds:

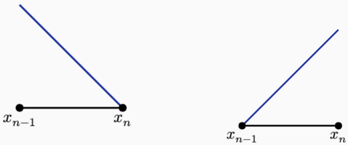
  Dirichlet: $u\big|_{\partial\Omega} = g_{\partial\Omega}$

  Neumann: $\left.\dfrac{\partial u}{\partial x}\right|_{\partial\Omega} = h_{\partial\Omega}$

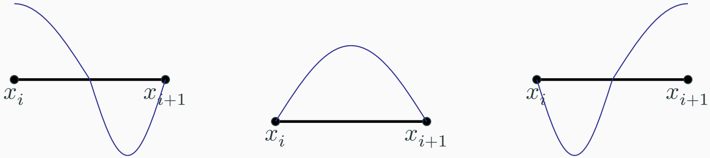
  Robin: $\alpha u\big|_{\partial\Omega} + \beta \left.\dfrac{\partial u}{\partial x}\right|_{\partial\Omega} = \ell_{\partial\Omega}$
- Galerkin Finite Element Formulation: Pick a finite dimensional $V_h \subseteq \mathring{H}^1(\Omega)$
- Resulting finite dimensional problem: $Au = b$

Examples of Basis Functions in 1d

Linear

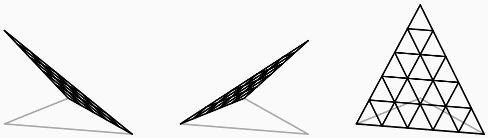


Quadratic

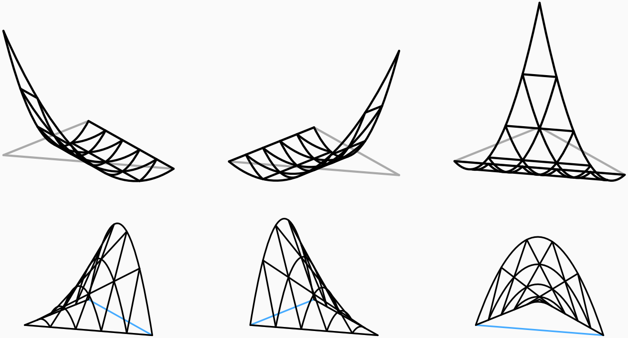


Examples of Basis Functions in 2d

Linear



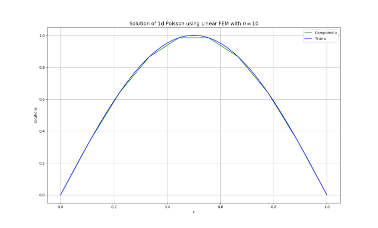
Quadratic

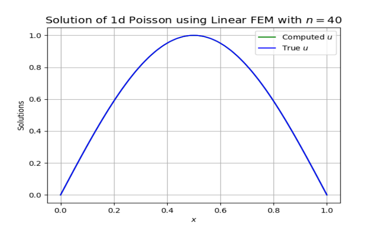


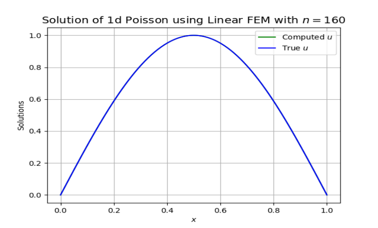
We sought to solve the following 1-D Poisson's Equation using FEM:

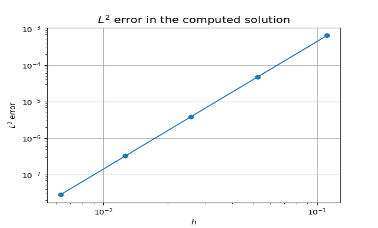
$$-u'' = -\pi^2 \sin(\pi x)$$

with homogeneous Dirichlet boundary conditions as u(0)=u(1)=0 where $\Omega = [0, 1]$









- Let $\mathcal{K}$ be a triangulation of Ω
- Let V_h be the space of continuous piecewise linears on $\mathcal{K}$
- From the variational formulation, we obtain the following finite element method: find $u_h \in V_{h,0}$ such that:

$$\int_\Omega \nabla u_h \cdot \nabla v \, dx = \int_\Omega f v dx, \quad \forall v \in V_{h,0} \tag{1}$$

- The finite element solution u_h defined by (1) then satisfies the estimate:

$$\|u - u_h\|_{L^2(\Omega)} = Ch^2 \|D^2 u\|_{L^2(\Omega)}$$

A Posteriori Error Estimate

- A posteriori error estimates for a computed solution provide insights into where the solution has a large error even if a true solution is not known
- It can thus be used for an adaptive refinement of a computed solution by modifying (usually by refinement) the triangulation where the error is large
- Now, the ∇u_h of the continuous piecewise linear finite element solution u_h is generally a discontinuous piecewise constant vector.
- Thus, when moving orthogonally across the boundary of one element to a neighbouring element, there is a jump in the normal derivative $n \cdot \nabla u_h$.
- This jump is denoted as $[n \cdot \nabla u_h]$ and plays a key role in a posteriori error analysis.
- The finite element solution u_h, defined by (1), satisfies the estimate:

$$|||u - u_h|||^2 \leq C \sum_{K \in \mathcal{K}} \eta_k^2(u_h)$$

- And the *element residual* $\eta_K(u_h)$ is defined by:

$$\eta_K(u_h) = h_K \|f + u_h\|_{L^2(K)} + \frac{1}{2} h_K^{1/2} \|[n \cdot \nabla u_h]\|_{L^2(\frac{\partial K}{\partial \Omega})}$$

- An adaptive finite element method (AFEM) is used in the numerical solution of partial differential equations by strategically improving the computational solution in an a posteriori manner by identifying subregions where the solution has a potentially large error.
- Adaptivity refers to use of a computational loop of the form:

$$\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE}$$

- One chooses to use a proxy indicator function for the error and in regions with a large error, we improve the solution approximation by either refining the underlying simplicial mesh or by raising the degree of a polynomial approximation to a suitably higher order.

Example of an a posteriori error estimate

Problem: $-u'' + u = f, \quad$ on $[0,1], \quad u(0) = u(1) = 0.$

A Posteriori Error Estimate: The finite element solution u_h for this problem will satisfy the following estimate:

$$||u - u_h||^2_{H^1} \leq C \sum_{i=1}^{n} h_i^2 ||f - u_h + u_h''||^2,$$

where the *element residual* $\eta(u_h)$ is given by:

$$\eta(u_h) = h_i ||f - u_h||_{H^1(I_i)}.$$

Future: We plan on working with developing and computing with a posteriori error estimators for the Poisson's equation in 2d and 3d in the immediate future in conjunction with higher order finite elements.

Thank You!