Character analogues of Ramanujan's identities

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Ramanujan's Entry

1. On page 253 in his lost notebook, *Ramanujan* recorded fascinating identities involving K-Bessel functions.

Entry 1

If α and β are any two positive numbers such that $\alpha\beta = \pi^2$ and ν is any complex number, then

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)n^{\nu/2}K_{\nu/2}(2n\alpha) - \sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)n^{\nu/2}K_{\nu/2}(2n\beta)$$
$$= \frac{1}{4}\Gamma\left(\frac{\nu}{2}\right)\zeta(\nu)\{\beta^{(1-\nu)/2} - \alpha^{(1-\nu)/2}\}$$
$$+ \frac{1}{4}\Gamma\left(-\frac{\nu}{2}\right)\zeta(-\nu)\{\beta^{(1+\nu)/2} - \alpha^{(1+\nu)/2}\}, \qquad (1)$$

where $\sigma_{-\nu}(n) = \sum_{d|n} d^{-\nu}$ and $K_\nu(z)$ denotes the modified Bessel function of order ν.

Koshliakov's Work

- Later, in 1955, *Guinand* derived a formula almost similar to (1) by appealing to a formula due to *Watson* involving the K-Bessel function.
- Letting $\nu \to 0$ in (1), we obtain

Koshliakov's identity

If α and β are any two positive numbers such that $\alpha\beta = \pi^2$, then

$$\sqrt{\alpha}\left(\frac{1}{4}\gamma - \frac{1}{4}\log(4\beta) + \sum_{n=1}^{\infty} d(n)K_0(2n\alpha)\right) = \sqrt{\beta}\left(\frac{1}{4}\gamma - \frac{1}{4}\log(4\alpha) + \sum_{n=1}^{\infty} d(n)K_0(2n\beta)\right), \tag{2}$$

- *Koshliakov*, in 1929, proved the formula (2) by employing Voronoï summation formula.

Voronoï's Work

Voronoï summation formula:

Let $f(x)$ is a function of bounded variation in (a, b) with $0 < a < b$, and $K_0(z)$ is modified bessel function of order 0, and $Y_\nu(z)$ denotes the Weber-Bessel function of order ν. Then

$$\sum_{a \leq n \leq b}{}' d(n) f(n) = \int_a^b (\log(x) + 2\gamma) f(x) dx + \sum_{n=1}^{\infty} d(n) \int_a^b f(x) \left(4K_0(4\pi\sqrt{nx}) - 2\pi Y_0(4\pi\sqrt{nx})\right) dx, \tag{3}$$

where γ is the Euler-Mascheroni constant.

- the prime $'$ on the summation of the left-hand side implies that if a or b is an integer, only $f(a)/2$ or $f(b)/2$ is counted, respectively.

Koshliakov's formula for twisted divisor sum

- In 2014, *B. C. Berndt, S. Kim and A. Zaharescu* studied character analogues of Koshliakov's formula (2) for even characters.
- They replaced the classical divisor function $d(n)$ with the twisted divisor sums, namely,

$$d_\chi(n) = \sum_{d|n} \chi(d), \qquad d_{\chi_1,\chi_2}(n) = \sum_{d|n} \chi_1(d)\chi_2(n/d),$$

where χ, χ_1 and χ_2 are the Dirichlet characters.

Let χ be a non-principal even primitive character mod q. Then for $\Re(z) > 0$

$$\frac{qL(1,\chi)}{4\tau(\chi)} + \sum_{n=1}^{\infty} d_\chi(n) K_0\left(\frac{2\pi nz}{\sqrt{q}}\right) = \frac{\sqrt{q}L(1,\chi)}{4z} + \frac{\tau(\chi)}{z\sqrt{q}} \sum_{n=1}^{\infty} d_{\bar{\chi}}(n) K_0\left(\frac{2\pi n}{z\sqrt{q}}\right)$$

where $K_0(z)$ is modified bessel function of order 0, and $Y_\nu(z)$ denotes the Weber-Bessel function of order ν and $\tau(\chi)$ is the Gauss sum.

Continued..

- For even real character χ, they established the positivity of $L(1,\chi)$, which is instrumental in proving Dirichlet's theorem on primes in arithmetic progressions.
- Later, *S. Kim* extended the definition of twisted divisor sums to twisted sums of divisor functions, namely,

$$\sigma_{k,\chi}(n) := \sum_{d|n} d^k \chi(d), \qquad \bar{\sigma}_{k,\chi}(n) := \sum_{d|n} d^k \chi(n/d),$$
$$\sigma_{k,\chi_1,\chi_2}(n) := \sum_{d|n} d^k \chi_1(d)\chi_2(n/d). \tag{4}$$

- They studied Riesz sum-type identities associated with them.
- Recently, *A. Dixit and A. Kesarwani* studied a new generalization of the modified Bessel function of the second kind. They derived a formula analogous to (1) associated with the generalized Bessel function.
- They proved that their formula is equivalent to the functional equation of a non-holomorphic Eisenstein series on $SL(2,\mathbb{Z})$.

Cohen's Work

- The study of the infinite series in (1) is of prime importance as it is intimately connected with the Fourier series expansion of non-holomorphic Eisenstein series on $SL(2,\mathbb{Z})$) or Maass wave forms.

Cohen's identity

For $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$ and any integer N such that $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, then

$$8\pi x^{\frac{\nu}{2}} \sum_{n=1}^{\infty} \sigma_{-\nu}(n) n^{\nu/2} K_{\nu}(4\pi\sqrt{nx}) = -\frac{\Gamma(\nu)\zeta(\nu)}{(2\pi)^{\nu-1}} + \frac{\Gamma(1+\nu)\zeta(1+\nu)}{\pi^{\nu+1}2^{\nu}x}$$
$$+\left\{ \frac{\zeta(\nu)x^{\nu-1}}{\sin\left(\frac{\pi\nu}{2}\right)} + \frac{2}{\sin\left(\frac{\pi\nu}{2}\right)} \sum_{j=1}^{N} \zeta(2j)\ \zeta(2j-\nu)x^{2j-1} \right.$$
$$\left. -\pi\frac{\zeta(\nu+1)x^{\nu}}{\cos(\frac{\pi\nu}{2})} + \frac{2}{\sin\left(\frac{\pi\nu}{2}\right)} \sum_{n=1}^{\infty} \sigma_{-\nu}(n) \frac{x^{2N+1}}{(n^2-x^2)} \left(n^{\nu-2N} - x^{\nu-2N}\right) \right\}. \quad (5)$$

B. C. Berndt, A. Dixit, A. Roy, and A. Zaharescu Work

- In 2017, B. C. Berndt, A. Dixit, A. Roy, and A. Zaharescu, in their seminal work, showed that Cohen-type identity (5) can be used to derive the Voronoï-type summation formula for $\sigma_s(n)$.

Voronoï summation formula for $\sigma_s(n)$

Let $0 < \alpha < \beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let f denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that $-\frac{1}{2} < \Re(\nu) < \frac{1}{2}$. Then

$$\sum_{\alpha<j<\beta} \sigma_{-\nu}(j) f(j) = \int_\alpha^\beta f(t) \left(\zeta(1-\nu, \chi)\ t^{-\nu} + \zeta(\nu+1)\right) dt$$

$$+ 2\pi \sum_{n=1}^{\infty} \sigma_{-\nu}(n) n^{\nu/2} \int_\alpha^\beta f(t)(t)^{-\frac{\nu}{2}} \left\{ \left(\frac{2}{\pi} K_\nu(4\pi\sqrt{nt}) - Y_\nu(4\pi\sqrt{nt}) \right) \cos\left(\frac{\pi\nu}{2}\right) - J_\nu(4\pi\sqrt{nt}) \sin\left(\frac{\pi\nu}{2}\right) \right\} dt.$$

Identites involving generalised divisor function

- Let us define the generalized divisor function

$$\sigma_z^{(r)}(n) = \sum_{d^r|n} d^z \tag{6}$$

- In 2022, *D. Banerjee and B.Maji* recently studied the infinite series involving the generalised divisor function and the modified K-Bessel functions.

Let $r \in \mathbb{Z}, z \in \mathbb{C}$ and a and x be any two positive real numbers,

$$\sum_{n=1}^{\infty} \sigma_z^{(r)}(n) n^{\frac{\nu}{2}} K_\nu(a\sqrt{nx}),$$

where ν is a complex number with $\Re(\nu) \geq 0$.

- It is important to note that $\sigma_z^{(1)}(n) = \sigma_z(n)$.
- Hence, almost all the Cohen-type identities can be derived from their results.

Identities associated with $\sigma_{\nu,\chi}(n)$ for odd characters

Thm 1 [Banerjee-K, Advance in Applied Mathematics, 2023]

Let k be an even, non-negative integer and χ be an odd primitive Dirichlet character modulo q. Then, for any $\Re(\nu) > 0$,

$$\sum_{n=1}^{\infty} \sigma_{k,\chi}(n) n^{\frac{\nu}{2}} K_\nu(a\sqrt{nx}) = \delta_k \frac{2^{\nu+1}}{a^{\nu+2}} \Gamma(1+\nu) L(1,\chi) x^{-\frac{\nu}{2}-1}$$
$$+ \frac{(-1)^{\frac{k}{2}} i q^k}{a^\nu 2^{k+2-\nu} \pi^{k+1}} \Gamma(\nu)\tau(\chi)\Gamma(k+1) L(k+1,\bar{\chi})\, x^{-\frac{\nu}{2}}$$
$$- \frac{(-1)^{\frac{k}{2}} i a^\nu q^{\nu+k}\ x^{\frac{\nu}{2}}}{2^{3\nu+k+2}\pi^{2\nu+k+1}} \Gamma(\nu+k+1)\tau(\chi) \sum_{n=1}^{\infty} \frac{\bar{\sigma}_{k,\bar{\chi}}(n)}{\left(n + \frac{a^2 q x}{16\pi^2}\right)^{\nu+k+1}},$$

where δ_k is given by

$$\delta_k = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k > 0. \end{cases} \tag{7}$$

The result corresponding to $\nu = 0$:

Theorem 2 [Banerjee-K, 2023]

Let k be an even, non-negative integer and χ be an odd primitive Dirichlet character modulo q. Then

$$\sum_{n=1}^{\infty} \sigma_{k,\chi}(n) K_0(a\sqrt{nx}) = \delta_k \frac{2}{a^2 x} L(1,\chi) - \frac{L(-k,\chi)}{4}\left(\log\left(\frac{8\pi}{a^2}\right) + \frac{L'(-k,\chi)}{L(-k,\chi)} - 2\gamma\right)$$
$$+ \frac{L(-k,\chi)}{4}\log x + (-1)^{\frac{k}{2}} \frac{ik!q^k}{2(2\pi)^{k+1}} \tau(\chi) \sum_{n=1}^{\infty} \bar{\sigma}_{k,\bar{\chi}}(n) \left(\frac{1}{n^{k+1}} - \frac{1}{(n + \frac{a^2 qx}{16\pi^2})^{k+1}}\right), \tag{8}$$

where δ_k is defined in (7).

Positivity of $L(1, \chi)$

Let us assume that χ is a real odd primitive Dirichlet character modulo q. Now setting $k = 0$ and then employing the functional equation in (8)

Result correspondence to $k = 0$

$$\sum_{n=1}^{\infty} d_\chi(n) K_0(a\sqrt{nx}) = \frac{L(1,\chi)}{x}\left(\frac{2}{a^2} - \frac{i\tau(\chi)}{4\pi} x \log x\right) - \frac{L(0,\chi)}{4}\left(\log\left(\frac{8\pi}{a^2}\right) + \frac{L'(0,\chi)}{L(0,\chi)} - 2\gamma\right) + \frac{ia^2 q\, x}{64\pi^3}\tau(\chi)\sum_{n=1}^{\infty}\frac{d_{\bar{\chi}}(n)}{n\left(n + \frac{a^2 qx}{16\pi^2}\right)}. \tag{9}$$

- Recall some facts on $K_0(x)$:

$$K_0(x) = \int_0^{\infty} e^{-x\cosh t}\, dt,$$

Positivity of $L(1,\chi)$ Continued.

1. Let us examine the left-hand side of (9).
   - From the integral representation of $K_0(x)$, we obtain $K_0(x)$ is positive and monotonically decreasing on the interval $(0,\infty)$.
   - The series representation of $K_0(x)$ is defined as

$$K_0(x) = -\log\left(\frac{x}{2}\right) I_0(x) + \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m}}{(m!)^2} \frac{\Gamma'(m+1)}{\Gamma(m+1)},$$

     where $I_0(x)$ is the Bessel function of the imaginary argument.
   - From its series representation mentioned above, one can infer that $K_0(x)$ tends to $+\infty$ as x decreases to 0.
   - Hence, the left-hand side of (9) approaches $+\infty$ as x decreases to 0.
2. Now, we examine the right-hand side of (9).
   - $i\tau(\chi)$ is real for real odd primitive Dirichlet character, we can easily deduce that the infinite series on the right-hand side of (9) tends to 0 as x decreases to 0.
   - $i\tau(\chi)$ is real and $x\log x$ tends to 0 as x decreases to 0, we infer that $\dfrac{L(1,\chi)}{x}$ tends to $+\infty$ as x decreases to 0,
3. Hence the strict positivity of $L(1,\chi)$ is proved.

Thank You!