Notions of isomorphism for reproducing kernel Hilbert spaces

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Introduction

Definition 1 : Let X be any non empty set, $\mathcal{F}(X, \mathbb{C})$ be collection of all functions from X to $\mathbb{C}$ and $\mathcal{H}$ be subset of $\mathcal{F}(X, \mathbb{C})$ such that

1. $\mathcal{H}$ is vector subspace of vector space $\mathcal{F}(X, \mathbb{C})$
2. $\mathcal{H}$ form a Hilbert space with the endowed inner product $\langle ., . \rangle$.
3. For every $x \in X$, Evaluation function $E_x : \mathcal{H} \to \mathbb{C}$ defined as $E_x(f) = f(x)$ is bounded.
   Then $\mathcal{H}$ is known as reproducing kernel Hilbert space on the set X.

Remark: Since by Riesz representation theorem $\forall\ x \in X\ \exists$ unique $k_x \in \mathcal{H}$ such that $E_x(f) = f(x) = \langle f, k_x \rangle$. Where k_x is known as reproducing kernel at a point x.

Introduction

Definition 2: An RKHS is a triplet $(X, \mathcal{H}, i)$ consisting of 3 objects:-

(1) A non-empty set X

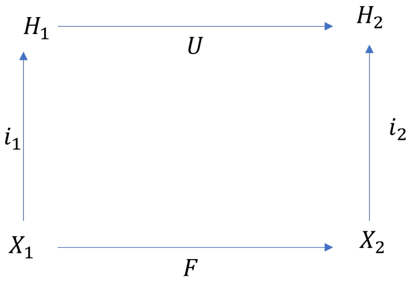
(2) A Hilbert space $\mathcal{H}$ consisting of functions from X to $\mathbb{C}$,and

(3) A function $i : X \to \mathcal{H}$ given by $i(x) = k_x$. Where k_x is a reproducing kernel at the point x.

Q When are two RKHSs considered to be 'same'?

Isomorphism between two RKHSs

Definition 3 : Let $\mathcal{H}_j, j = 1, 2$ be two Hilbert function spaces on the sets $X_j,\ j = 1, 2$ with reproducing kernels $K_j(y, x) = k_y^j(x),\ j = 1, 2.$ Then $(X_1, \mathcal{H}_1, i_1)$ is 'same' as $(X_2, \mathcal{H}_2, i_2)$ if $\exists$ a bijection $F : X_1 \to X_2$ and a unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that the diagram below commutes.

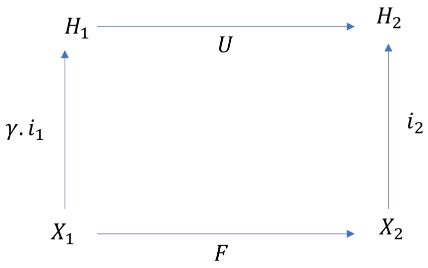


Isomorphism between two RKHSs

Theorem 1: Let $(X_1, \mathcal{H}_1, i_1)$ and $(X_2, \mathcal{H}_2, i_2)$ be two Hilbert function spaces then TFSAE:-

1. $\exists$ a bijective map $F : X_1 \to X_2$ and a unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$ that maps for each $y \in X_1$. The one dimensional subspace $\mathbb{C}k_y^1 \subseteq \mathcal{H}_1$ onto $\mathbb{C}k_y^2 \subseteq \mathcal{H}_2$.
2. $\exists$ a bijection $F : X_1 \to X_2$ and a nowwhere vanishing complex valued function $\gamma : X_1 \to \mathbb{C}$ such that for every $y \in X_1$, The mapping $k_y^1 \to \frac{1}{\gamma(y)} k_{F(y)}^2$ extends to a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$.
3. ($\mathcal{H}_2$ **is a rescaling of** $\mathcal{H}_1$) $\exists$ a bijection $F : X_1 \to X_2$ and a nowwhere vanishing complex valued function $\gamma : X_1 \to \mathbb{C}$ such that
$$\forall\ x, y \in X_1, K_2(F(x), F(y)) = \overline{\gamma(x)}\gamma(y)K_1(x, y).$$
4. ($\mathcal{H}_1$ **is isometrically isomorphic to** $\mathcal{H}_2$) $\exists$ a bijection $F : X_1 \to X_2$ and a nowwhere vanishing complex valued function $\gamma : X_1 \to \mathbb{C}$ such that diagram below commutes.

Isomorphism between two RKHSs



where $\gamma.i_1 : X_1 \to \mathbb{C}$ defined as $(\gamma.i_1)(x) = \gamma(x)i_1(x) = \gamma(x)k_x^1$.

Isomorphism between two RKHSs

Definition 3: An isomorphism of reproducing kernel Hilbert spaces from H_1 to $\mathcal{H}_2$(or simply an RKHS isomorphism) is a bijective bounded linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ defined by

$$T(k_x^1) = \gamma(x) K_{F(x)}^2, x \in X_1$$

where $\gamma : X_1 \to C$ is a nowhere-vanishing function and $F : X_1 \to X_2$ is a bijection.

Multiplier algebra of a RKHS

Definition 4:Let $\mathcal{H}_j,\ j = 1, 2$ be reproducing kernel Hilbert spaces on the some set X and let $K_j,\ j = 1, 2$ denote their kernel functions. A function $f : X \to \mathbb{C}$ is called a multiplier of $\mathcal{H}_1$ into $\mathcal{H}_2$ if $f\mathcal{H}_1 \subseteq \mathcal{H}_2$ where $f\mathcal{H}_1 = \{fh; h \in \mathcal{H}_1\}$.

$\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ denote the set of all multiplier of $\mathcal{H}_1$ onto $\mathcal{H}_2$.

Preposition : Let $\mathcal{H}$ be an RKHS on X with kernel K and Let $f : X \to \mathbb{C}$ be a function, Let $\mathcal{H}_0 = \{h : fh = 0\}$ and let $\mathcal{H}_1 = \mathcal{H}_0^{\perp}$. Set $\mathcal{H}_f = f\mathcal{H} = f\mathcal{H}_1$ and define an inner product on $\mathcal{H}_f$ by

$$\langle fh_1, fh_2 \rangle = \langle h_1, h_2 \rangle \text{ for } h_1, h_2 \in \mathcal{H}_1.$$

Then $\mathcal{H}_f$ is an RKHS on X with kernel, $K_f(x, y) = f(x)K(x, y)\overline{f(y)}$.

Isomorophism between Multiplier algebras

Definition 5: We define a " **Multiplier Algebra Isomorphism**" between multiplier algebras $\mathcal{M}(\mathcal{H}_1)$ and $\mathcal{M}(\mathcal{H}_2)$, to be a **complete isomorphism** $\phi : \mathcal{M}(\mathcal{H}_1) \to \mathcal{M}(\mathcal{H}_2)$ that is implemented as

$$\phi(f) = f \circ G \, , \quad f \in \mathcal{M}(H_1)$$

where $G : X_1 \to X_2$ is a bijection.
If such an isomorphism exists then $\mathcal{M}(\mathcal{H}_1)$ and $\mathcal{M}(\mathcal{H}_2)$ are isomorphic as a multiplier algebras.
If ϕ is a **completely isometric** then we say that $\mathcal{M}(\mathcal{H}_1)$ and $\mathcal{M}(\mathcal{H}_2)$ are completely **isometrically isomorphic** as multiplier algebras.

Isomorphism between Multiplier algebras

Theorem 2:Let $d \in \mathbb{N} \cup \{\infty\}$, X and Y be two finite subsets of $B_d = \{x \in \mathbb{C}^d : \|x\| < 1\}$.Then following statements are equivalent :-

(i) $\mathcal{H}_x$ and $\mathcal{H}_y$ are isomorphic as RKHSs (where $\mathcal{H}_x = H^2_d|_X$ and $\mathcal{H}_y = H^2_d|_Y$)

(ii) $\mathcal{M}(\mathcal{H}_x)$ and $\mathcal{M}(\mathcal{H}_y)$ are isomorphic as multiplier algebras.

(iii) $\text{Card(X)} = \text{Card(Y)}$

References

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2. Pick Interpolation and Hilbert Function Spaces by *Jim Agler, John E. McCarthy*.
3. Distance between reproducing kernel Hilbert spaces and geometry of finite sets in the unit ball by *Danny Ofek, Satish K. Pandey, Orr Moshe Shalit.*

THANK YOU