Higher moments of averages of Ramanujan sums

Shivani Goel
(Joint work with Prof. M. Ram Murty)

IIIT Delhi

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Ramanujan Sums

- The Ramanujan sum is defined by:

$$c_q(n) = \sum_{\substack{j=0 \\ (j,q)=1}}^{q-1} e\left(\frac{nj}{q}\right)$$

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- It is an integer-valued function. Moreover

$$c_q(n) = \sum_{\substack{d|n \\ d|q}} d\mu\left(\frac{q}{d}\right),$$

  where $\mu(n)$ is the Möbius function defined as

$$\mu(n) = \begin{cases} (-1)^r & n = p_1 \cdots p_r, \\ 0 & \text{otherwise.} \end{cases}$$

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- It is a multiplicative function of q.

Ramanujan Expansion

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2. If r(n) denotes the number of ways of writing n as a sum of two squares, then

$$r(n) = \pi \sum_{q \geq 1} \frac{(-1)^q}{2q+1} c_{2q+1}(n).$$

Orthogonality property

Theorem (Carmichael 1932)

Ramanujan sums satisfy an orthogonality property:

$$\frac{1}{q}\sum_{n=1}^{q} c_{q_1}(n)c_{q_2}(n) = \begin{cases} \phi(q) & \text{if } q_1 = q_2 = q, \\ 0 & \text{otherwise.} \end{cases}$$

Distribution of Ramanujan sums

The k^{th} moments of the Ramanujan sums is defined by

$$S_k(x, y) = \sum_{n \le y} \left(\sum_{q \le x} c_q(n) \right)^k .$$

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Theorem (T. H. Chan, A. V. Kumchev, 2012)

Let x be a large real number and $y \ge x$, then

$$S_1(x,y) = y - \frac{x^2}{4\zeta(2)} + O(xy^{1/3}\log x + x^3 y^{-1}).$$

Moment of Ramanujan sums

Theorem (T. H. Chan, A. V. Kumchev, 2012)

Let x be a large real number, $y \geq x$, and $B > 0$ be fixed.

- *If $y > x^2(\log x)^B$, then*

$$S_2(x,y) = \frac{yx^2}{2\zeta(2)} + O(x^4 + xy\log x).$$

- *if $x \leq y \leq x^2(\log x)^B$, then*

$$S_2(x,y) = \frac{yx^2}{2\zeta(2)}(1 + 2\kappa(u)) + O(x^2 y \log^{10} x(x^{-1/2} + (\frac{y}{x})^{-1/2})),$$

where $u = log(yx^{-2})$ and $\kappa(u)$ is the a certain Fourier integral and satisfies the inequalities $\kappa(u) > -0.4$, $\kappa(u) << exp(-|u|^{3/5-\epsilon})$. In particular, $\kappa(u) = o(1)$, as $|u| \to \infty$.

Generalized Ramanujan sums

Cohen generalized the Ramanujan Sums as:

$$c_{q,b}(n) = \sum_{\substack{j=0 \\ (j,q^b)_b=1}}^{q^b-1} e\left(\frac{nj}{q^b}\right) = \sum_{\substack{d^b|n \\ d|q}} d^b \mu\left(\frac{q}{d^b}\right),$$

where $(j, q^b)_b$ is the b^{th} power gcd of j and q^b.

$$(j, q^b)_b = \max\{d^b : d^b|j \text{ and } d^b|q^b\}.$$

Moment of generalized Ramanujan sums

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$$S_{k,b}(x,y) = \sum_{n \leq y} \left(\sum_{q \leq x} c_{q,b}(n) \right)^k .$$

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Theorem (N. Robles, A. Roy, 2017)

Let $y \geq x^{3b/2} log^5 x$, then for $b = 1, 2$.

$$S_{1,b}(x,y) = y - \frac{x^{1+b}}{2(1+b)\zeta(1+b)} + O(x^b y^{1/3} log^4 y + x^{2b+1} y^{-2/3} + x^{b+1} y^{-1/3})$$

and for $b \geq 3$,

$$S_{1,b}(x,y) = y + O(x^b y^{1/3} log^4 y),$$

Moment of generalized Ramanujan sums

Theorem (N. Robles, A. Roy, 2017)

For $b = 1, 2$, and $x^{2b} < y < x^{2b+b^2} log^{5/2(b+1)} x$,

$$S_{2,b}(x, y) = \frac{yx^{1+b}}{(1+b)\zeta(1+b)} - \frac{x^{2+2b}}{2(1+b)^2\zeta^2(1+b)} + O(y^{-1}x^{2+4b} + x^{2b+1}(\log x + \log\log x) + x^{2b}y^{1/3+1/6b} \log^5 x \log\log x(\log^4 x + \log^4 \log x) + yx^{1/2+b}(\log^3 x + \log^3 \log x)).$$

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For $b > 2$ and $y > x^{3b/2}$,

$$S_{2,b}(x,y) = \frac{yx^{1+b}}{(1+b)\zeta(1+b)} + O(x^{2b}y^{1/3+1/6b} \log^5 x \log\log x(\log^4 x + \log^4 \log x) + yx^{1/2+b}(\log^3 x + \log^3 \log x)).$$

k^{th} moment of generalized Ramanujan sums

Proposition (N. Robles, A. Roy, 2017)

Let k,b be the two positive integers and $y > x^{k(b+1)} \log^{k+1} x$, *then*

$$S_{k,b}(x, y) = A_{k,b}(x, y) + O(x^{k(b+1)} \log^{k} x),$$

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Higher moments of Ramanujan sums

Theorem (G., M. R. Murty)

For $k \geq 3$ and $y > x^k$, as $x \to \infty$, we have

$$S_k(x, y) = yx^k Q(\log x) + O\left(yx^{k-\theta}\right),$$

where $Q \in \mathbb{R}[X]$ is a polynomial of exact degree $2^k - 2k - 1$ and $0 \leq \theta \leq 1$.

Idea of proof

- From the definition

$$S_k(x,y) = \sum_{n\le y}\left(\sum_{q\le x} c_q(n)\right)^k = \sum_{n\le y}\sum_{q_1,\cdots,q_k\le x}\sum_{\substack{d_1|q_1\\ d_1|n}} d_1\mu\left(\frac{q_1}{d_1}\right)\cdots\sum_{\substack{d_k|q_k\\ d_k|n}} d_k\mu\left(\frac{q_k}{d_k}\right)$$

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$$= \sum_{q_1,\cdots,q_k\le x}\ \sum_{d_1|q_1,\cdots,d_k|q_k} d_1\cdots d_k\mu\left(\frac{q_1}{d_1}\right)\cdots\mu\left(\frac{q_k}{d_k}\right)\left(\frac{y}{[d_1,\cdots,d_k]} + O(1)\right)$$

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$$= y\sum_{q_1,\cdots,q_k\le x}\sum_{d_1|q_1,\cdots,d_k|q_k}\frac{d_1\cdots d_k}{[d_1,\cdots,d_k]}\mu\left(\frac{q_1}{d_1}\right)\cdots\mu\left(\frac{q_k}{d_k}\right)+O\left(x^{2k}\right).$$

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- Next, the Brèteche Tauberian theorem gives the required result.

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where $Q(\log x)$ is a polynomial of exact degree $2^k - 2k - 1$ and $0 \leq \theta \leq 1$.

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Thank You!